Chapter 8

FITTED TENSION SPLINE APPROXIMATION METHOD FOR SOLVING SYSTEM OF SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS

Tarun Gupta

Anshul Dubey

Department of Mathematics, IFTM University, Moradabad, Uttar Pradesh-244102

Corresponding author E-mail: mathtarun26@gmail.com

Abstract

A fitted tension spline approximation method is proposed for solving a coupled system of singularly perturbed delay differential equations. The proposed method employs a cubic spline in tension on a uniform grid to construct the difference scheme. The method has been shown to consistently converge, regardless of the perturbation parameter, as confirmed by numerical testing.

Keywords: Convection-diffusion, delay term, singular perturbation.

1. Introduction

Singularly perturbed delay differential equations (SPDDEs) are a class of differential equations that incorporate both small perturbation parameters and time delays. These equations are widely used in mathematical modelling across various fields, such as: Human pupil-light reflex [1],HIV infection [2]-[3],Biological oscillators [4], Control theory [5], Neuronal activation [6],Physiological processes [7]-[8], Bistable devices in electronics [9], Population dynamics [10].These differential equations arise when the future behavior of the system is influenced not only by its current state but also by its past history.

Over the past twenty years, significant research has been conducted on numerical methods for SPDEs. While effective numerical techniques have been developed for single SPDDEs, there are only a limited number of results available in the literature for systems of such equations. Subburayan and

Ramanujam [11]-[12] came up with two approaches: the initial value technique and the asymptotic numerical method: to tackle convection-diffusion and reaction-diffusion equations. Meanwhile, Selvi and Ramanujam [13] proposed an iterative numerical method tailored for a coupled system of singularly perturbed equations.

Here, we derived a fitted tension spline approximation method to solve systems of SPDDEs. Traditional methods tend to stumble when ε gets tiny compared to the grid width h used in discretization. Our goal isto prove that cubic spline in tension can deliver solid, accurate results whether ε is small or large relative to h. Tension splines were first introduced by Schweikert [14] to reduce spurious oscillations that often occur in cubic spline curve fitting. This concept was later explored and developed further by researchers such as Pruess [15], de Boor [16], and others.

In developing ε-uniform methods, one effective approach is the fitted operator method. This technique was initially proposed by Allen et al. [17] for modelling viscous fluid flow past a cylinder. A comprehensive overview of ε-uniform fitted operator methods can be found in the work of Doolan et al. [18]. Further contributions were made by Kadalbajoo and Sharma [19], who applied an ε-uniform fitted operator method to boundary value problems involving singularly perturbed delay differential equations.

The objective of the present study is to construct an ε-uniform numerical scheme for solving boundary value problems arising from coupled systems of singularly perturbed convection–diffusion delay differential equations. To achieve this, we employ a fitted operator method in conjunction with cubic splines in tension to effectively handle such complex problems.

2. Statement of the Problem

Consider the following coupled system of SPDDEs of convection-diffusion type:

$$\begin{cases} -\varepsilon\, w_1''(x) + p_1(x)w_1'(x) + \sum_{k=1}^{2} q_{1k}(x)w_k(x) + \sum_{k=1}^{2} r_{1k}(x)w_k(x-1) = f_1(x), \;\; x \in \Omega \\ -\varepsilon\, w_2''(x) + p_2(x)w_2'(x) + \sum_{k=1}^{2} q_{2k}(x)w_k(x) + \sum_{k=1}^{2} r_{2k}(x)w_k(x-1) = f_2(x), \;\; x \in \Omega \\ \qquad w_1(x) = \phi_1(x), \quad x\epsilon[-1,0], \quad w_1(2) = l_1, \\ \qquad w_2(x) = \phi_2(x), \quad x\epsilon[-1,0], \quad w_2(2) = l_2, \end{cases}$$

(1)

where$0 < \varepsilon \ll 1$,

$$p_i(x) \geq \alpha_i \geq \alpha > 0, \qquad i = 1,2$$

$$q_{11}(x) > 0, \qquad q_{22}(x) > 0, \qquad q_{12}(x) \leq 0, \qquad q_{21}(x) \leq 0,$$

$$q_{i1}(x) + q_{i2}(x) \geq \beta_i \geq \beta > 0, \qquad i = 1,2,$$

$$r_{ij}(x) \leq 0, \quad i = 1,2, \qquad j = 1,2,$$

$$-{-}\gamma \leq -\gamma_i \leq r_{i1}(x) + \ r_{i2}(x) < 0, \ i = 1,2, \ \beta - \gamma > 0,$$

thefunction$p_i, q_{ik}, r_{ik}, f_i \ \in C^4(\Omega), i = 1,2, k = 1,2, \Omega = (0,2), \overline{\Omega} = [0,2], \Omega^- = (0,1), \Omega^+ = (1,2)$ and $\phi_i, i = 1,2$ are smooth functions on [-1,0]. It may be noted that problem (1) exhibits a strong boundary layer at x=2.

3. Derivation of Method

Tension spline approximation method is derived on a uniform mesh as follows:

Let$h$ is step size and $x_0 = 0, x_{2N} = 2, x_i = ih, i = 1 \ to \ 2N - 1$.

The functions $S_j(x,\tau) = S_j(x), j = 1,2$ satisfying the following equations:

$$S_j''(x) - \tau S_j(x) = [S_j''(x_i) - \ \tau S_j(x_i)]\frac{(x_{i+1} - x)}{h} + [S_j''(x_{i+1}) - \ \tau S_j(x_{i+1})]\frac{(x - x_i)}{h},$$

$$x \in [x_i, x_{i+1}]$$

(2) where, $S_j(x_i) = W_j(x_i) \simeq \ w_j(x_i), j = 1,2$ and $\tau > 0$ is termed as tension factor.

Solving Eq. (2), we get

$$S_j(x) = C_j e^{\frac{\mu x}{h}} + D_j e^{-\frac{\mu x}{h}} + (\frac{M_{j,i} - \tau W_{j,i}}{\tau})(\frac{x - x_{i+1}}{h}) + (\frac{M_{j,i+1} - \tau W_{j,i+1}}{\tau})(\frac{x_i - x}{h}),$$

where$C_j$and $D_j$are the arbitrary constants, whose values are found with the use of interpolatory conditions $S_j(x_{i+1}) = W_{j,i+1}, S_j(x_i) = W_{j,i}$ for $j = 1,2$.

Takeμ=$h\tau^{\frac{1}{2}}$and $M_{j,i} = S_j''(x_i)$, we get

$$S_j(x)=\frac{h^2}{\mu^2\sinh\mu}\left[M_{j,i+1}\sinh\frac{\mu(x-x_i)}{h}+M_{j,i}\sinh\frac{\mu(x_{i+1}-x)}{h}\right]$$
$$-\frac{h^2}{\mu^2}\left[\frac{(x-x_i)}{h}\left(M_{j,i+1}-\frac{\mu^2}{h^2}W_{j,i+1}\right)\right.$$
$$\left.+\frac{(x_{i+1}-x)}{h}\left(M_{j,i}-\frac{\mu^2}{h^2}W_{j,i}\right)\right]$$

(3) Differentiating Eq. (3) and taking $x\to x_i$ we obtain

$$S_j'(x_i^+)=\frac{\left(W_{j,i+1}-W_{j,i}\right)}{h}-\frac{h}{\mu^2}\left[\left(1-\frac{\mu}{\sinh\mu}\right)M_{j,i+1}-(1-\mu\coth\mu)M_{j,i}\right].$$

Considering the interval $(x_{i-1}-x_i)$and proceeding similarly, we get

$$S_j'(x_i^-)=\frac{\left(W_{j,i}-W_{j,i-1}\right)}{h}+\frac{h}{\mu^2}\left[-(1-\mu\coth\mu)M_{j,i}+\left(1-\frac{\mu}{\sinh\mu}\right)M_{j,i-1}\right]$$

Equating the left-hand and right-hand derivatives at x_i, we have

$$\frac{\left(W_{j,i+1}-W_{j,i}\right)}{h}-\frac{h}{\mu^2}\left[\left(1-\frac{\mu}{\sinh\mu}\right)M_{j,i+1}-(1-\mu\coth\mu)M_{j,i}\right]$$
$$=\frac{\left(W_{j,i}-W_{j,i-1}\right)}{h}$$
$$+\frac{h}{\mu^2}\left[-(1-\mu\coth\mu)M_{j,i}+\left(1-\frac{\mu}{\sinh\mu}\right)M_{j,i-1}\right]$$

(4) Thus, we get a tridiagonal system

$$h^2(\mu_1M_{j,i-1}+2\mu_2M_{j,i}+\mu_1M_{j,i+1}=\ W_{j,i+1}-2W_{j,i}+W_{j,i-1},i=1\ to\ 2N-1$$

(5) Forj $=1,2,$ where $\mu_1=\frac{1}{\mu^2}\left(1-\frac{\mu}{\sinh\mu}\right),\mu_2$=$\frac{1}{\mu^2}(\mu\coth\mu-1)$, and $M_{j,i}=S_j''(x_i)$,

$$i=1\ to\ 2N-1.$$

The equation (5) is consistent if$\mu_1+\mu_2=\frac{1}{2}$.

From the boundary conditions $W_{j,i}=\phi_{j,i},-N\le i\le 0,W_{j,2N}=l_j$, where $\phi_{j,i}=\phi_j(x_i)$.

Take the notation

$p_1(x_i) = p_{1,i}, p_2(x_i) = p_{2,i}, q_{1j}(x_i) = q_{1j,i}, q_{2j}(x_i) = q_{2j,i}, r_{1j}(x_i) = r_{1ji}, r_{2j}(x_i) = r_{2ji}$ and $f_j(x_i) = f_{j,i}$.

From Eq. (1), we have

$\varepsilon M_{1,k} = p_{1,k}W'_{1,k} + q_{11,k}W_{1,k} + q_{12,k}W_{2,k} + r_{11,k}W_1(x_k - 1) + r_{12,k}W_2(x_k - 1) - f_{1,k}$,

$\varepsilon M_{2,k} = p_{2,k}W'_{2,k} + q_{21,k}W_{1,k} + q_{22,k}W_{2,k} + r_{21,k}W_1(x_k - 1) + r_{22,k}W_2(x_k - 1) - f_{2,k}$,

Substituting $M_{1,k}$ and $M_{2,k}$with $k = i, i \pm 1$ and

$$W'_{j,i} = \frac{W_{j,i+1} - W_{j,i-1}}{2h}, \;\; j = 1,2,$$

$$W'_{j,i+1} = \frac{3W_{j,i+1} - 4W_{j,i} + W_{j,i-1}}{2h}, \;\; j = 1,2,$$

$$W'_{j,i-1} = \frac{-W_{j,i+1} + 4W_{j,i} - W_{j,i-1}}{2h}, \;\; j = 1,2.$$

In Eq. (5), we obtain the following system of linear equations in$W_{1,i}$ and $W_{2,i}$,

$$\begin{aligned}\{-\varepsilon - 1.5\mu_1 hp_{1,i-1} + \mu_1 h^2 q_{11,i-1} - \mu_2 hp_{1,i} + 0.5\mu_1 hp_{1,i+1})W_{1,i-1} + (2\varepsilon \\ + 2\mu_1 hp_{1,i-1} + 2\mu_2 h^2 q_{11,i} - 2\mu_1 hp_{1,i+1})W_{1,i} + (-\varepsilon \\ - 0.5\mu_1 hp_{1,i-1} + \mu_2 hp_{1,i} + 1.5\mu_1 hp_{1,i+1} + \mu_1 h^2 q_{11,i+1})W_{1,i+1} \\ + h^2(\mu_1 q_{12,i-1}W_{2,i-1} + 2\mu_2 q_{12,i}W_{2,i} + \mu_1 q_{12,i+1}W_{2,i+1}) \\ = h^2[\{\mu_1 f_{1,i-1} + 2\mu_2 f_{1,i} + \mu_1 f_{1,i+1}\} \\ - \{\mu_1 r_{11,i-1}W_1(x_{i-1-N}) + 2\mu_2 r_{11,i}W_1(x_{i-N}) \\ + \mu_1 r_{11,i+1}W_1(x_{i+1-N})\} - \{\mu_1 r_{12,i-1}W_2(x_{i-1-N}) \\ + 2\mu_2 r_{12,i}W_2(x_{i-N}) + \mu_1 r_{12,i+1}W_2(x_{i+1-N})\}\end{aligned}$$

$$\{(-\varepsilon - 1.5\mu_1 hp_{2,i-1} + \mu_1 h^2 q_{22,i-1} - \mu_2 hp_{2,i} + 0.5\mu_1 hp_{2,i+1})W_{2,i-1} + (2\varepsilon + 2\mu_1 hp_{2,i-1} + 2\mu_2 h^2 q_{22,i} - 2\mu_1 hp_{2,i+1})W_{2,i} + (-\varepsilon - 0.5\mu_1 hp_{2,i-1} + \mu_2 hp_{2,i} + 1.5\mu_1 hp_{2,i+1} + \mu_1 h^2 q_{22,i+1})W_{2,i+1} + h^2(\mu_1 q_{21,i-1} W_{1,i-1} + 2\mu_2 q_{21,i} W_{1,i} + \mu_1 q_{21,i+1} W_{1,i+1}) = h^2[\{\mu_1 f_{2,i-1} + 2\mu_2 f_{2,i} + \mu_1 f_{2,i+1}\} - \{\mu_1 r_{22,i-1} W_2(x_{i-1-N}) + 2\mu_2 r_{22,i} W_2(x_{i-N}) + \mu_1 r_{22,i+1} W_2(x_{i+1-N})\} - \{\mu_1 r_{21,i-1} W_1(x_{i-1-N}) + 2\mu_2 r_{21,i} W_1(x_{i-N}) + \mu_1 r_{21,i+1} W_1(x_{i+1-N})\}]$$

For $i = 1\ to\ 2N - 1$

(6) Incorporating a fitting factor in Eq. (6), we get

$$\{(-\varepsilon\sigma_1 - 1.5\mu_1 hp_{1,i-1} + \mu_1 h^2 q_{11,i-1} - \mu_2 hp_{1,i} + 0.5\mu_1 hp_{1,i+1})W_{1,i-1} + \left(2\varepsilon\sigma_1 + 2\mu_1 hp_{1,i-1} + 2\mu_2 h^2 q_{11,i} - 2\mu_1 hp_{1,i+1}\right)W_{1,i} + \left(-\varepsilon\sigma_1 - 0.5\mu_1 hp_{1,i-1} + \mu_2 hp_{1,i} + 1.5\mu_1 hp_{1,i+1} + \mu_1 h^2 q_{11,i+1}\right)W_{1,i+1} + h^2\left(\mu_1 q_{12,i-1} W_{2,i-1} + 2\mu_2 q_{12,i} W_{2,i} + \mu_1 q_{12,i+1} W_{2,i+1}\right) = h^2[\{\mu_1 f_{1,i-1} + 2\mu_2 f_{1,i} + \mu_1 f_{1,i+1}\} - \{\mu_1 r_{11,i-1} W_1(x_{i-1-N}) + 2\mu_2 r_{11,i} W_1(x_{i-N}) + \mu_1 r_{11,i+1} W_1(x_{i+1-N})\} - \{\mu_1 r_{12,i-1} W_2(x_{i-1-N}) + 2\mu_2 r_{12,i} W_2(x_{i-N}) + \mu_1 r_{12,i+1} W_2(x_{i+1-N})\}],$$

$$\{(-\varepsilon\sigma_2 - 1.5\mu_1 hp_{2,i-1} + \mu_1 h^2 q_{22,i-1} - \mu_2 hp_{2,i} + 0.5\mu_1 hp_{2,i+1})W_{2,i-1} + (2\varepsilon\sigma_2 + 2\mu_1 hp_{2,i-1} + 2\mu_2 h^2 q_{22,i} - 2\mu_1 hp_{2,i+1})W_{2,i} + (-\varepsilon\sigma_2 - 0.5\mu_1 hp_{2,i-1} + \mu_2 hp_{2,i} + 1.5\mu_1 hp_{2,i+1} + \mu_1 h^2 q_{22,i+1})W_{2,i+1} + h^2(\mu_1 q_{21,i-1} W_{1,i-1} + 2\mu_2 q_{21,i} W_{1,i} + \mu_1 q_{21,i+1} W_{1,i+1}) = h^2[\{\mu_1 f_{2,i-1} + 2\mu_2 f_{2,i} + \mu_1 f_{2,i+1}\} - \{\mu_1 r_{22,i-1} W_2(x_{i-1-N}) + 2\mu_2 r_{22,i} W_2(x_{i-N}) + \mu_1 r_{22,i+1} W_2(x_{i+1-N})\} - \{\mu_1 r_{21,i-1} W_1(x_{i-1-N}) + 2\mu_2 r_{21,i} W_1(x_{i-N}) + \mu_1 r_{21,i+1} W_1(x_{i+1-N})\}]$$

(7)

where

$$\sigma_j = \frac{p_j(x)\frac{h}{\varepsilon}}{2}\coth\left(\frac{p_j(x)\frac{h}{\varepsilon}}{2}\right), \qquad j = 1, 2.$$

We solved the above system by taking $\mu_1 = \frac{1}{12}, \mu_2 = \frac{5}{12}$.

4. Numerical Examples

The maximum absolute pointwise errors using the double mesh principle [18] is given by

$$E_{i,\varepsilon}^M = \max_{0\le j\le M}\left|W_{i,j}^M - W_{i,2j}^{2M}\right|, \qquad i = 1,2.$$

The ε - uniformmaximum absolute error is given by

$$E_i^M = \max_{\varepsilon} E_{i,\varepsilon}^M, \quad i = 1,2.$$

The numerical rate of convergence is given by

$$R_i^M = \frac{\log\left(E_i^M/E_i^{2M}\right)}{\log 2}, i = 1,2.$$

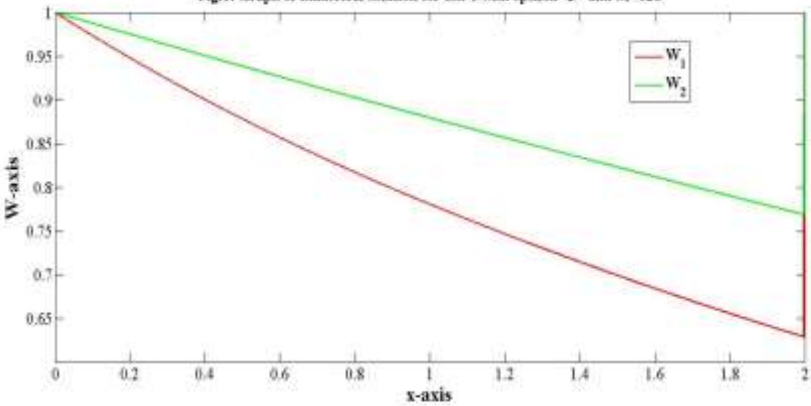
Example 1:

$$-\varepsilon\, w_1''(x) + 11w_1'(x) + 6w_1(x) - 2w_2(x) - w_1(x-1) = 0$$
$$-\varepsilon\, w_2''(x) + 16w_2'(x) - 2w_1(x) + 5w_2(x) - 5w_2(x-1) = 0$$
$$w_1(x) = 1, if -1 \le x \le 0, w_1(2) = 1$$
$$w_2(x) = 1, if -1 \le x \le 0, w_2(2) = 1$$

Table 1:

$M \rightarrow$	64	128	256	512	1024	2048
E_1^M	5.7306e-04	2.8882e-04	1.4499e-04	7.2644e-05	3.6358e-05	1.8188e-05
R_1^M	0.9885	0.9942	0.9971	0.9985	0.9992	-
E_2^M	1.2714e-04	6.5319e-05	3.3098e-05	1.6659e-05	8.3570e-06	4.1854e-06
R_2^M	0.9609	0.9807	0.9904	0.9952	0.9976	-

Fig.1: Graph of numerical solution for Ex. 1 with epsilon=2^{-8} and M=128



Example 2:

$$-\varepsilon\, w_1''(x) + 11w_1'(x) + 10w_1(x) - 2w_2(x) + x^2 w_1(x-1) - x w_2(x-1) = e^{x}$$

$$-\varepsilon\, w_2''(x) + 16w_2'(x) - 2w_1(x) + 10w_2(x) - x w_1(x-1) - x w_2(x-1) = e^{x^2}$$

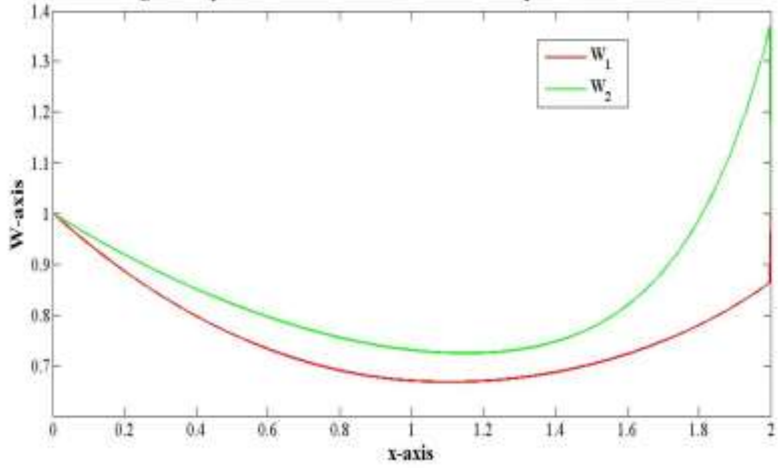
$$w_1(x) = 1, if -1 \le x \le 0, w_1(2) = 1$$

$$w_2(x) = 1, if -1 \le x \le 0, w_2(2) = 1$$

Table 2:

$M \rightarrow$	64	128	256	512	1024	2048
E_1^M	5.2975e-03	2.7382e-03	1.3925e-03	7.0229e-04	3.5266e-04	1.7671e-04
R_1^M	0.9521	0.9755	0.9876	0.9938	0.9969	-
E_2^M	2.0714e-02	1.0705e-02	5.4430e-03	2.7444e-03	1.3779e-03	6.9043e-04
R_2^M	0.9522	0.9759	0.9879	0.9939	0.9969	-

Fig. 2: Graph of numerical solution for Ex.2 with epsilon=2^{-8} and M=128



5. Conclusion

We have proposed a uniform mesh difference scheme using fitted tension spline approximation method that converges consistently. It's designed for a coupled system of SPDDEs of the convection-diffusion type. We've included numerical examples to highlight how well the scheme performs. The results show that our fitted tension spline approximation method delivers oscillation-free solutions for $0 < \varepsilon < 1$ across the entire domain, $0 < x < 2$. We tested it on two examples with varying ε values.

Disclaimer (Artificial Intelligence): No AI tool has been used to generate data and design any image. All data have been taken from well-reputed published journals and the language is manually modified without using any software.

Competing Interests: Authors have declared that no competing interests exist.

References

[1] Longtin, A., Milton, G. J., Complex oscillations in the human pupil light reflex with mixed and delayed feedback, Mathematical Biosciences, 90 (1988), 183-199.

[2] Culshaw, R. V., Ruan, S., A delay differential equation model of HIV infection of CD4 + T-cells, Mathematical Biosciences, 165 (1) (2000), 27-39.

[3] Nelson, P. W., Perelson, A. S., Mathematical analysis of delay differential equation models of HIV 1 infection, Mathematical Biosciences, 179 (2002), 73-94.

[4] Derstine, M.W., Gibbs, H. M., Hopf, F.A., Kaplan, D.L., Bifurcation gap in a hybrid optical system, Phys. Rev. A 26 (1982), 3720-3722.

[5] Glizer, V. Y., Asymptotic solution of a boundary-value problem for linear singularly-perturbed functional differential equations arising in optimal control theory, Journal of Optimization Theory and Applications, 106 (2000), 49-85.

[6] Wilbur, W.J., Rinzel, J., An analysis of Steins model for stochastic neuronal excitation, Biological Cybernetics, 45 (1982), 107-114.

[7] Mayer, H., Zaenker, K.S., an der Heiden, U., A basic mathematical model of immune response, Chaos, 5 (1995), 155-161.

[8] Mackey, M.C., Glass, L., Oscillation and chaos in physiological control systems, Science, 197 (1977), 287-289.

[9] Chen, Y., Wu, J., The asymptotic shapes of periodic solutions of a singular delay differential system, Journal of Differential Equations, 169 (2001), 614-632.

[10] Kuang, Y., Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

[11] Subburayan, V., Ramanujam,N., An initial value method for singularly perturbed system of reaction-diffusion type delay differential equations, Journal of the Korea Society for Industrial and Applied Mathematics, 17(4) (2013), 221-237

[12] Subburayan, V., Ramanujam,N., An asymptotic numerical method for singularly perturbed weakly coupled system of convection- diffusion type differential difference equations, Novi Sad J. Math., 44(2), (2014), 53-68.

[13] Selvi P. Avudai, Ramanujam, N., An Iterative numerical method for a weakly coupled system of singularly perturbed convection diffusion equations with negative shifts, Int. J. Appl. Comput. Math. 3 (2017), 147-160.

[14] Schweikert D.G., An Interpolation Curve Using a Spline in Tension, J. Math. Phys., 45 (1966), 312-317.

[15] Pruess S., Properties of splines in tension, Journal of Approximation Theory, 17 (1976), 86-96.

[16] Boor C.De., A practical guide to splines, Applied Mathematical Science Series 27, Springer, New York, 1978.

[17] De G Allen D. N., Southwell R. V., Relaxation methods applied to determine the motion in 2D, of a viscous fluid past a fixed cylinder. Q J Mech. Appl. Math. 8(2)(1955),129–145.

[18] Miller J.J.H., E. O. Riordan, I. G., Shishkin, fitted numerical methods for singular perturbation problems, Word Scientific, Singapore,1996.
[19] Kadalbajoo M. K., Sharma K. K., An -uniform fitted operator method for solving boundary-value problems for singularly perturbed delay differential equations: layer behavior. Int. J.Comput. Math. 80(10)(2003),1261–1276.