SPLINE COMPRESSION TECHNIQUE FOR SYSTEM OF SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS

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Abstract:
In this work, we develop a spline compression technique for the numerical solution of system of singularly perturbed delay differential equations (SPDDEs). The method constructs a piecewise cubic spline in compression to approximate the solution, effectively balancing accuracy in boundary layers with computational efficiency. Numerical experiments on benchmark problems validate the effectiveness of the proposed technique, showing that it yields highly accurate results even for very small perturbation parameters and significant delays.
Keywords: Compression Spline, Delay Term, Singular Perturbation, Convection-Diffusion.

Introduction:

A more realistic model must account for how a system is influenced not only by its present state but also by its past and even anticipated future states. Therefore, real-world systems are often described using differential equations with delays or advances. Such equations play an important role in mathematical modelling across many disciplines, including Human pupil-light reflex [1], HIV infection [2, 3], Biological oscillators [4], Control systems [5], Neuronal activity [6], Physiological processes [7, 8], Bistable electronic devices [9], Population dynamics [10]. They appear in situations where the system's evolution is determined by present values together with different influences from its previous states.

Over the past twenty years, significant research has been conducted on numerical methods for SPDDEs. While effective numerical techniques have been developed for single SPDDEs, there are only a limited number of results available in the literature for systems of such equations. Subburayan and Ramanujam [11, 12] came up with two approaches: the initial value technique and the asymptotic numerical method: to tackle convection-diffusion and reaction-diffusion equations. Meanwhile, Selvi and Ramanujam [13] proposed an iterative numerical method tailored for a coupled system of SPDDEs.

Here, we derived a fitted compression spline approximation scheme to solve systems of SPDDEs. Traditional methods tend to stumble when ε (perturbation parameter) gets tiny compared to the grid width h used in discretization. Our goal is to prove that cubic spline in compression can deliver solid, accurate results whether ε is small or large relative to h. Splines techniques were first introduced by Schweikert [14] to reduce spurious oscillations that often occur in cubic spline curve fitting. This concept was later explored and developed further by researchers such as Pruess [15], de Boor [16], and others.

In developing ε-uniform methods, one effective approach is the fitted operator method. This technique was initially proposed by Allen *et al.* [17] for modelling viscous fluid flow past a cylinder. A comprehensive overview of ε-uniform fitted operator methods can be found in the work of Miller and Riordan [18]. Further contributions were made by Kadalbajoo and Sharma [19], who applied an ε-uniform fitted operator method to boundary value problems involving singularly perturbed delay differential equations.

The objective of the present study is to construct an ε-uniform numerical scheme for solving boundary value problems arising from coupled systems of SPDDEs. To achieve this, we employ a fitted operator method in conjunction with cubic splines in compression to effectively handle such complex problems.

Statement of the Problem:

Consider the following system of SPDDEs of convection-diffusion type:

$$\begin{cases} -\varepsilon\, v_1''(x) + p_1(x)v_1'(x) + \sum_{k=1}^{2} q_{1k}(x)v_k(x) + \sum_{k=1}^{2} r_{1k}(x)v_k(x-1) = f_1(x), x \in \Omega \\ -\varepsilon\, v_2''(x) + p_2(x)v_2'(x) + \sum_{k=1}^{2} q_{2k}(x)v_k(x) + \sum_{k=1}^{2} r_{2k}(x)v_k(x-1) = f_2(x), x \in \Omega \\ v_1(x) = \phi_1(x), x\epsilon[-1,0], v_1(2) = l_1, \\ v_2(x) = \phi_2(x), x\epsilon[-1,0], v_2(2) = l_2, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$, the function $p_i, q_{ik}, r_{ik}, f_i \in C^4(\Omega), i = 1,2, k = 1,2, \Omega = (0,2), \overline{\Omega} = [0,2]$, $\Omega^- = (0,1), \Omega^+ = (1,2)$ and $\phi_i, i = 1,2$ are smooth functions on [-1,0]. It may be noted that problem (1) exhibits a strong boundary layer at x=2.

SPLINE COMPRESSION APPROXIMATION DIFFERENCE SCHEME:

Spline compression approximation difference scheme is developed on a uniform mesh as follows:

Let h is step size and $x_0 = 0, x_{2N} = 2, x_i = ih, i = 1\ to\ 2N-1$.

The functions $S_j(x,\tau) = S_j(x), j = 1,2$ satisfying the following differential equations:

$$S_j''(x) + \tau S_j(x) = [S_j''(x_i) + \tau S_j(x_i)]\,\frac{(x_{i+1}-x)}{h} + [S_j''(x_{i+1}) - \tau S_j(x_{i+1})]\,\frac{(x-x_i)}{h},$$

$x \in [x_i, x_{i+1}]$ (2)

where, $S_j(x_i) = V_j(x_i) \simeq v_j(x_i), j = 1, 2$ and $\tau > 0$ is termed as compression factor.

Solving Eq. (2), we get

$S_j(x) = C_j \cos\frac{\mu x}{h} + D_j \sin\frac{\mu x}{h} + (\frac{M_{j,i}+\tau V_{j,i}}{\tau})\ (\frac{x_{i+1}-x}{h}) + (\frac{M_{j,i+1}+\tau V_{j,i+1}}{\tau})\ (\frac{x-x_i}{h})$,

where C_j and D_j are the arbitrary constants, whose values are found with the use of interpolatory conditions $S_j(x_{i+1}) = V_{j,i+1}, S_j(x_i) = V_{j,i}$ for $j = 1, 2$.

Take μ=$h\ \tau^{\frac{1}{2}}$ and $M_{j,i} = S_j''(x_i)$, we get

$$S_j(x) = -\frac{h^2}{\mu^2 \sin\mu}\left[M_{j,i+1}\sin\frac{\mu(x-x_i)}{h} + M_{j,i}\sin\frac{\mu(x_{i+1}-x)}{h}\right] + \frac{h^2}{\mu^2}\left[\frac{(x-x_i)}{h}\left(M_{j,i+1} + \frac{\mu^2}{h^2}V_{j,i+1}\right) + \frac{(x_{i+1}-x)}{h}\left(M_{j,i} + \frac{\mu^2}{h^2}V_{j,i}\right)\right]$$

(3)

Differentiating Eq. (3) and taking $x \to x_i$ we obtain

$$S_j'(x_i^+) = \frac{\left(V_{j,i+1} - V_{j,i}\right)}{h} + \frac{h}{\mu^2}\left[\left(1 - \frac{\mu}{\sin\mu}\right)M_{j,i+1} - (1 - \mu\cot\mu)M_{j,i}\right].$$

Considering the interval (x_{i-1}, x_i) and proceeding similarly, we get

$$S_j'(x_i^-) = \frac{\left(V_{j,i} - V_{j,i-1}\right)}{h} + \frac{h}{\mu^2}\left[(1 - \mu\cot\mu)M_{j,i} - \left(1 - \frac{\mu}{\sin\mu}\right)M_{j,i-1}\right]$$

Equating the left-hand and right-hand derivatives at x_i, we have

$$\frac{\left(V_{j,i+1} - V_{j,i}\right)}{h} + \frac{h}{\mu^2}\left[\left(1 - \frac{\mu}{\sin\mu}\right)M_{j,i+1} - (1 - \mu\cot\mu)M_{j,i}\right] = \frac{\left(V_{j,i} - V_{j,i-1}\right)}{h} + \frac{h}{\mu^2}\left[(1 - \mu\cot\mu)M_{j,i} - \left(1 - \frac{\mu}{\sin\mu}\right)M_{j,i-1}\right]$$

(4)

Thus, we get a tridiagonal system

$$h^2(\mu_1 M_{j,i-1} + 2\mu_2 M_{j,i} + \mu_1 M_{j,i+1} = V_{j,i+1} - 2V_{j,i} + V_{j,i-1}, i = 1\ to\ 2N - 1 \quad (5)$$

For j $= 1, 2$, where $\mu_1 = \frac{1}{\mu^2}\left(\frac{\mu}{\sin\mu} - 1\right), \mu_2 = \frac{1}{\mu^2}(1 - \mu\cot\mu)$, and $M_{j,i} = S_j''(x_i)$,

$$i = 1\ to\ 2N - 1.$$

The equation (5) is consistent if $\mu_1 + \mu_2 = \frac{1}{2}$.

From the boundary conditions $V_{j,i} = \phi_{j,i}, -N \le i \le 0, V_{j,2N} = l_j$, where $\phi_{j,i} = \phi_j(x_i)$.

Take the notation

$p_1(x_i) = p_{1,i}, p_2(x_i) = p_{2,i},\ q_{1j}(x_i) = q_{1j,i},\ q_{2j}(x_i) = q_{2j,i}, r_{1j}(x_i) = r_{1ji},\ r_{2j}(x_i) = r_{2ji}$ and $f_j(x_i) = f_{j,i}$.

From Eq. (1), we have

$\varepsilon M_{1,k} = p_{1,k}V'_{1,k} + q_{11,k}V_{1,k} + q_{12,k}V_{2,k} + r_{11,k}V_1(x_k - 1) + r_{12,k}V_2(x_k - 1) - f_{1,k}$,

$\varepsilon M_{2,k} = p_{2,k}V'_{2,k} + q_{21,k}V_{1,k} + q_{22,k}V_{2,k} + r_{21,k}V_1(x_k - 1) + r_{22,k}V_2(x_k - 1) - f_{2,k}$,

Substituting $M_{1,k}$ and $M_{2,k}$ with $k = i, i \pm 1$ and

$$V'_{j,i} = \frac{V_{j,i+1} - V_{j,i-1}}{2h}, j = 1, 2,$$

$$V'_{j,i+1} = \frac{3V_{j,i+1} - 4V_{j,i} + V_{j,i-1}}{2h}, j = 1, 2,$$

$$V'_{j,i-1} = \frac{-V_{j,i+1} + 4V_{j,i} - 3V_{j,i-1}}{2h}, j = 1, 2.$$

In Eq. (5), we obtain the following system of linear equations in $V_{1,i}$ and $V_{2,i}$,

$$\begin{aligned}\{-\varepsilon - 1.5\mu_1 hp_{1,i-1} + \mu_1 h^2 q_{11,i-1} - \mu_2 hp_{1,i} + 0.5\mu_1 hp_{1,i+1})V_{1,i-1} + (2\varepsilon + 2\mu_1 hp_{1,i-1} \\ + 2\mu_2 h^2 q_{11,i} - 2\mu_1 hp_{1,i+1})V_{1,i} + (-\varepsilon - 0.5\mu_1 hp_{1,i-1} + \mu_2 hp_{1,i} + 1.5\mu_1 hp_{1,i+1} \\ + \mu_1 h^2 q_{11,i+1})V_{1,i+1} + h^2(\mu_1 q_{12,i-1}V_{2,i-1} + 2\mu_2 q_{12,i}V_{2,i} + \mu_1 q_{12,i+1}V_{2,i+1}) \\ = h^2[\{\mu_1 f_{1,i-1} + 2\mu_2 f_{1,i} + \mu_1 f_{1,i+1}\} \\ - \{\mu_1 r_{11,i-1}V_1(x_{i-1-N}) + 2\mu_2 r_{11,i}V_1(x_{i-N}) + \mu_1 r_{11,i+1}V_1(x_{i+1-N})\} \\ - \{\mu_1 r_{12,i-1}V_2(x_{i-1-N}) + 2\mu_2 r_{12,i}V_2(x_{i-N}) + \mu_1 r_{12,i+1}V_2(x_{i+1-N})\}\end{aligned}$$

$$\begin{aligned}\{(-\varepsilon - 1.5\mu_1 hp_{2,i-1} + \mu_1 h^2 q_{22,i-1} - \mu_2 hp_{2,i} + 0.5\mu_1 hp_{2,i+1})V_{2,i-1} + (2\varepsilon + 2\mu_1 hp_{2,i-1} \\ + 2\mu_2 h^2 q_{22,i} - 2\mu_1 hp_{2,i+1})V_{2,i} + (-\varepsilon - 0.5\mu_1 hp_{2,i-1} + \mu_2 hp_{2,i} + 1.5\mu_1 hp_{2,i+1} \\ + \mu_1 h^2 q_{22,i+1})V_{2,i+1} + h^2(\mu_1 q_{21,i-1}V_{1,i-1} + 2\mu_2 q_{21,i}V_{1,i} + \mu_1 q_{21,i+1}V_{1,i+1}) \\ = h^2[\{\mu_1 f_{2,i-1} + 2\mu_2 f_{2,i} + \mu_1 f_{2,i+1}\} \\ - \{\mu_1 r_{22,i-1}V_2(x_{i-1-N}) + 2\mu_2 r_{22,i}V_2(x_{i-N}) + \mu_1 r_{22,i+1}V_2(x_{i+1-N})\} \\ - \{\mu_1 r_{21,i-1}V_1(x_{i-1-N}) + 2\mu_2 r_{21,i}V_1(x_{i-N}) + \mu_1 r_{21,i+1}V_1(x_{i+1-N})\}]\end{aligned}$$

For $i = 1\ to\ 2N - 1$

(6)

Incorporating a fitting factor in Eq. (6), we get

$$\{(-\varepsilon\sigma_1 - 1.5\mu_1 hp_{1,i-1} + \mu_1 h^2 q_{11,i-1} - \mu_2 hp_{1,i} + 0.5\mu_1 hp_{1,i+1})V_{1,i-1}$$
$$+ \left(2\varepsilon\sigma_1 + 2\mu_1 hp_{1,i-1} + 2\mu_2 h^2 q_{11,i} - 2\mu_1 hp_{1,i+1}\right)V_{1,i}$$
$$+ \left(-\varepsilon\sigma_1 - 0.5\mu_1 hp_{1,i-1} + \mu_2 hp_{1,i} + 1.5\mu_1 hp_{1,i+1} + \mu_1 h^2 q_{11,i+1}\right)V_{1,i+1}$$
$$+ h^2\left(\mu_1 q_{12,i-1}V_{2,i-1} + 2\mu_2 q_{12,i}V_{2,i} + \mu_1 q_{12,i+1}V_{2,i+1}\right)$$
$$= h^2[\{\mu_1 f_{1,i-1} + 2\mu_2 f_{1,i} + \mu_1 f_{1,i+1}\}$$
$$- \{\mu_1 r_{11,i-1}V_1(x_{i-1-N}) + 2\mu_2 r_{11,i}V_1(x_{i-N}) + \mu_1 r_{11,i+1}V_1(x_{i+1-N})\}$$
$$- \{\mu_1 r_{12,i-1}V_2(x_{i-1-N}) + 2\mu_2 r_{12,i}V_2(x_{i-N}) + \mu_1 r_{12,i+1}V_2(x_{i+1-N})\}],$$

$$\{(-\varepsilon\sigma_2 - 1.5\mu_1 hp_{2,i-1} + \mu_1 h^2 q_{22,i-1} - \mu_2 hp_{2,i} + 0.5\mu_1 hp_{2,i+1})V_{2,i-1} + (2\varepsilon\sigma_2 + 2\mu_1 hp_{2,i-1}$$
$$+ 2\mu_2 h^2 q_{22,i} - 2\mu_1 hp_{2,i+1})V_{2,i} + (-\varepsilon\sigma_2 - 0.5\mu_1 hp_{2,i-1} + \mu_2 hp_{2,i}$$
$$+ 1.5\mu_1 hp_{2,i+1} + \mu_1 h^2 q_{22,i+1})V_{2,i+1} + h^2(\mu_1 q_{21,i-1}V_{1,i-1} + 2\mu_2 q_{21,i}V_{1,i}$$
$$+ \mu_1 q_{21,i+1}V_{1,i+1})$$
$$= h^2[\{\mu_1 f_{2,i-1} + 2\mu_2 f_{2,i} + \mu_1 f_{2,i+1}\}$$
$$- \{\mu_1 r_{22,i-1}V_2(x_{i-1-N}) + 2\mu_2 r_{22,i}V_2(x_{i-N}) + \mu_1 r_{22,i+1}V_2(x_{i+1-N})\}$$
$$- \{\mu_1 r_{21,i-1}V_1(x_{i-1-N}) + 2\mu_2 r_{21,i}V_1(x_{i-N}) + \mu_1 r_{21,i+1}V_1(x_{i+1-N})\}]$$

(7)

where

$$\sigma_j = \frac{p_j(x)\frac{h}{\varepsilon}}{2}\coth\left(\frac{p_j(x)\frac{h}{\varepsilon}}{2}\right), j = 1, 2.$$

We solved the above system by taking $\mu_1 = \frac{1}{18}, \mu_2 = \frac{4}{9}$.

Numerical Examples:

The maximum absolute pointwise errors using the double mesh principle is given by

$$E_{i,\varepsilon}^M = \max_{0\le j\le M}\left|V_{i,j}^M - V_{i,2j}^{2M}\right|, i = 1, 2.$$

The ε - uniform maximum absolute error is given by

$$E_i^M = \max_{\varepsilon} E_{i,\varepsilon}^M, i = 1, 2.$$

The numerical rate of convergence is given by

$$R_i^M = \frac{\log\left(E_i^M / E_i^{2M}\right)}{\log 2}, i = 1, 2.$$

Example 1:

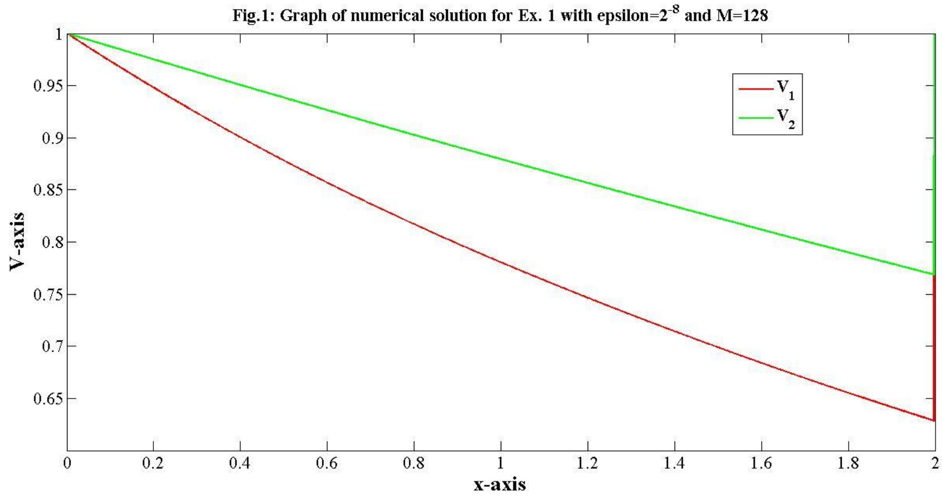
$$-\varepsilon\, v_1''(x) + 11v_1'(x) + 6v_1(x) - 2v_2(x) - v_1(x-1) = 0$$
$$-\varepsilon\, v_2''(x) + 16v_2'(x) - 2v_1(x) + 5v_2(x) - v_2(x-1) = 0$$
$$v_1(x) = 1, if -1 \le x \le 0, v_1(2) = 1$$

$$v_2(x) = 1, if -1 \leq x \leq 0, v_2(2) = 1$$

Table 1:

$M \rightarrow$	64	128	256	512	1024	2048
E_1^M	5.7950e-04	2.9202e-04	1.4657e-04	7.3424e-05	3.6746e-05	1.8382e-05
R_1^M	0.9887	0.9944	0.9972	0.9986	0.9993	-
E_2^M	1.5066e-04	7.7020e-05	3.8934e-05	1.9573e-05	9.8133e-06	4.9133e-06
R_2^M	0.9680	0.9841	0.9921	0.9960	0.9980	-

Fig.1: Graph of numerical solution for Ex. 1 with epsilon=2^{-8} and M=128



Example 2:

$$-\varepsilon\, v_1''(x) + 11v_1'(x) + 10v_1(x) - 2v_2(x) + x^2 v_1(x-1) - xv_2(x-1) = e^x$$

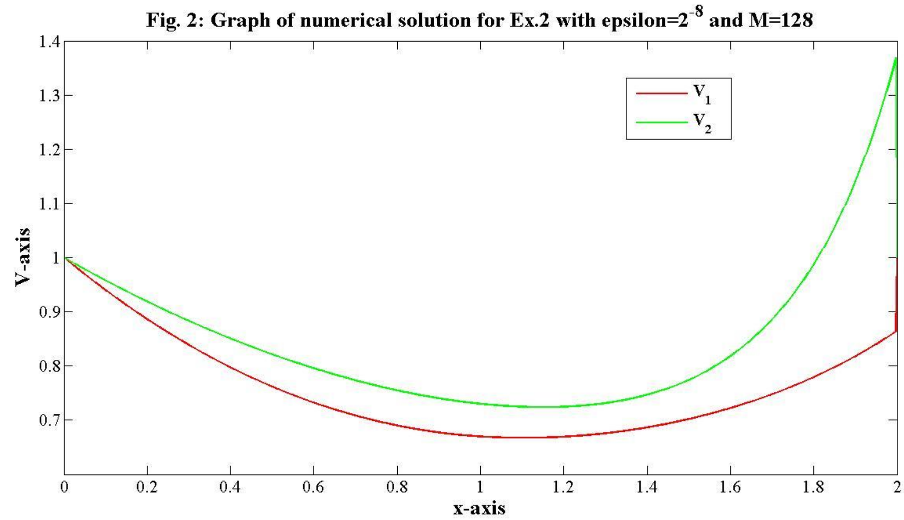
$$-\varepsilon\, v_2''(x) + 16v_2'(x) - 2v_1(x) + 10v_2(x) - xv_1(x-1) - xv_2(x-1) = e^{x^2}$$

$$v_1(x) = 1, if -1 \leq x \leq 0, v_1(2) = 1$$

$$v_2(x) = 1, if -1 \leq x \leq 0, v_2(2) = 1$$

Table 2:

$M \rightarrow$	64	128	256	512	1024	2048
E_1^M	5.3221e-03	2.7533e-03	1.4009e-03	7.0667e-04	3.5491e-04	1.7785e-04
R_1^M	0.9508	0.9748	0.9872	0.9935	0.9967	-
E_2^M	2.0320e-02	1.0560e-02	5.3835e-03	2.7181e-03	1.3657e-03	6.8452e-04
R_2^M	0.9443	0.9719	0.9859	0.9929	0.9964	-



Interpretation:

We have proposed a uniform mesh difference scheme using fitted compression spline approximation method that converges consistently. It's designed for a coupled system of SPDDEs of the convection-diffusion type. We've included numerical examples to highlight how well the scheme performs. The results show that our fitted tension spline approximation method delivers oscillation-free solutions for $0 < \varepsilon < 1$ across the entire domain, $0 < x < 2$. We tested it on two examples with varying ε values.

References:

1. Longtin, A., Milton, G. J. (1988). Complex oscillations in the human pupil light reflex with mixed and delayed feedback, *Mathematical Biosciences*, 90, 183-199.
2. Culshaw, R. V., Ruan, S. (2000). A delay differential equation model of HIV infection of CD4 + T-cells, *Mathematical Biosciences*, 165 (1), 27-39.
3. Nelson, P. W., Perelson, A. S. (2002). Mathematical analysis of delay differential equation models of HIV 1 infection, *Mathematical Biosciences*, 179, 73-94.
4. Derstine, M.W., Gibbs, H. M., Hopf, F.A., Kaplan, D.L. (1982). Bifurcation gap in a hybrid optical system, *Physical Review A,* 26, 3720-3722.
5. Glizer, V. Y. (2000). Asymptotic solution of a boundary-value problem for linear singularly-perturbed functional differential equations arising in optimal control theory, *Journal of Optimization Theory and Applications*, 106, 49-85.
6. Wilbur, W.J., Rinzel, J. (1982). An analysis of Steins model for stochastic neuronal excitation, *Biological Cybernetics*, 45, 107-114.
7. Mayer, H., Zaenker, K. S., an der Heiden, U. (1995). A basic mathematical model of immune response, *Chaos*, 5, 155-161.

8. Mackey, M.C., Glass, L. (1977). Oscillation and chaos in physiological control systems, *Science*, 197, 287-289.
9. Chen, Y., Wu, J. (2001). The asymptotic shapes of periodic solutions of a singular delay differential system, *Journal of Differential Equations*, 169, 614-632.
10. Kuang, Y. (1993). Delay differential equations with applications in population dynamics, *Academic Press, New York*.
11. Subburayan, V., Ramanujam, N. (2013). An initial value method for singularly perturbed system of reaction-diffusion type delay differential equations, *Journal of the Korea Society for Industrial and Applied Mathematics*, 17(4), 221-237
12. Subburayan, V., Ramanujam, N. (2014). An asymptotic numerical method for singularly perturbed weakly coupled system of convection- diffusion type differential difference equations, *Novi Sad J. Math.,* 44(2), 53-68.
13. Selvi P. Avudai, Ramanujam, N. (2017). An Iterative numerical method for a weakly coupled system of singularly perturbed convection-diffusion equations with negative shifts, *International Journal of Applied and Computational Mathematics*, 3, 147-160.
14. Schweikert, D.G. (1966). An Interpolation curve using a spline in tension, *Journal of Mathematical Physics*, 45, 312-317.
15. Pruess, S. (1976). Properties of splines in tension, *Journal of Approximation Theory*, 17, 86-96.
16. Boor C. De. (1978). A practical guide to splines, *Applied Mathematical Science* Series 27, Springer, New York.
17. De G Allen D. N., Southwell R. V. (1955). Relaxation methods applied to determine the motion in 2D, of a viscous fluid past a fixed cylinder. *The Quarterly Journal of Mechanics and Applied Mathematics,* 8(2), 129–145.
18. Miller, J.J.H., E. O. Riordan, I. G. (1996). Shishkin, fitted numerical methods for singular perturbation problems, *Word Scientific, Singapore.*
19. Kadalbajoo, M. K., Sharma, K. K. (2003). An ε-uniform fitted operator method for solving boundary-value problems for singularly perturbed delay differential equations: layer behavior. *International Journal of Computer Mathematics,* 80(10), 1261–1276.