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STOCHASTIC ANALYSIS OF A MULTI STATE SYSTEM (MSS) HAVING WEIGHTED 3-OUT OF *N*: D UNIT USING COPULA

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Abstract
In the present paper authors have modeled a multi state system having three subsystems A, B and C connected in series. Subsystem A is a controller unit whose failure results into the complete failure of the system. Subsystem B has *m* identical units arranged in series configuration. Further, subsystem C is a weighted 3-out of *n*: D system having *n* units. Subsystem C consists of two units namely ordinary and key having weights 1 and 2 respectively. The repair facility will repair the failed units in two cases, firstly if the system is in complete breakdown state and second when the system is working but three successive failures have occurred. Also whenever two key units fail, two repair facilities will repair the failed components simultaneously with different rates. The model has been solved with the help of Supplementary variable technique, Laplace transformation and copula. The transition state probabilities, asymptotic behaviour and some characteristics of the system such as reliability, availability, M.T.T.F., cost effectiveness of system reliability with respect to different parameters have been obtained. At last the proposed model is demonstrated via numerical examples.

Keywords: Reliability, availability, M.T.T.F., weighted *k*-out of *n*: D systems, Gumbel-Hougaard copula.

1.1 Introduction and description of model
Technical equipment and its operating conditions are becoming highly complicated now days. Increasingly critical functions [7, 10] are assigned to technical apparatus in production and control. Every system possesses its own unique characteristics. For example, some types of apparatus have hundreds and thousands of components but other may have a single component, some have identical components while other consists of non identical components etc. Similarly we can have systems with multi states. The reliability of a multi state system (MSS) [5] is a recently emerging field at the junction of traditional binary reliability and performance analysis. A MSS can be defined as a system that can have a finite number of discrete performance rates from perfect functioning to complete failure, resulting from the degradation or/and failure of some elements in the system. Such a MSS is usually viewed as a failure state if its performance rate falls below the user demand. These different performance levels of a system may occur due to critical structural arrangement of units inside the system. Further a review of past literatures [3, 9] show that various types of system configuration such as *k*-out-of-*n* and circular consecutive systems etc. have recently been studied. But most of these studies concern with the systems in which the units are sharing an equal fraction of total load [4]. In real world a system with non identical components may have units bearing different part of total load. A weighted system [1, 2] possesses this type of characteristic. It can be defined as a system in which a weight is associated with every unit of the system. This weight may be equal or it may be the case that each unit has a different weight. Generally speaking, a weighted *k*-out of-*n*: F system consists of *n* units each of which is associated with a weight $w_i > 0$, $i = 1, 2, \ldots, n$. These system work as long as the sum of

weights of failed components is less than a certain threshold value k and fail if sum exceeds k. For any weighted system the vector w = (w_1, w_2,...,w_1) called the weight vector and the threshold value k are fixed. For instance, a lighting system having a fixed number of light bulbs with different performance characteristics such as different wattage and illumination power is a weighted system. When a certain level of illumination is deemed minimally acceptable, one can assume this value as the threshold value beyond which the system is defined as failed.

Considering all the above aspects in the present paper, the authors have considered a multi state system having three discrete states: operational, degraded and failed. System considered consists of three subsystems A, B and C arranged in series. It is assumed that the units are stochastically, economically and structurally independent to each other. So the condition of components does not influence the life of other components. The cost of joint maintenance of a group of components is equal to the total cost of individual maintenance and each component structurally forms an entity that is not further subdivided for reliability study. Here subsystem A is a controller unit, if it fails the system fails. Since it has been assumed that subsystem A can suffer from retrogressive failure i.e. the risk of failure is highest in the initial period for subsystem A, so it can fail from the fully operational state. In subsystem B there are m identical units arranged in series. The subsystem B fails if any of the m units fails. The main concern of this paper is to study weighted systems. Here the third subsystem i.e. subsystem C is a weighted 3-out of-n: D system. The weight vector associated with this subsystem is w = (1, 2, 1, 2,….1, 2) and the threshold value for the subsystem C and hence the whole system to be in degraded state is 3. The subsystem C will be in degraded state if out of n units, the weight of failed units is 3. Further, the threshold value for the failure of the subsystem C and hence for the whole system is 4. The system will be repaired if (i) the system is in failed state or (ii) three successive failures occur in the system. Also if more than two key components are in failure mode two different repair facilities will repair the failed units simultaneously with different repair rates. The joint probability distribution for this case is obtained with the help of Gumbel-Hougaard family of copula [6, 8]. Failure rates are assumed to be constant in general whereas the repair rates are assumed to be a function of time. The purpose of this paper is to accomplish following objectives with the help of Supplementary variable technique, Laplace transformation and copula methodology.

(1) To evaluate the transition state probabilities of the system.

(2) To determine the asymptotic behaviour of the system.

(3) To calculate various measures such as reliability, availability, M.T.T.F. and cost effectiveness of the system.

The paper is organized as follows:
In the following section 2 assumptions regarding the system, the state specification chart and the transition state diagram are presented. Then the next section describes the nomenclature. Next to that the equations corresponding to system and their solution have been determined and at last numerical examples and conclusions are given.

(1.1a) Copula

A two dimensional copula is a function C: [0, 1] × [0, 1] → [0, 1] that satisfies following two properties.
1. Boundary conditions:
(a) For all t in [0, 1], $C(t, 0) = C(0, t) = 0$
(b) For all t in [0, 1], $C(t, 1) = C(1, t) = t$

2. Rectangular inequality: If u_1, u_2, v_1, v_2 are in [0, 1] with $u_1 \le u_2$ and $v_1 \le v_2$, then

$$C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \ge 0$$

(1.1b) Gumbel-Hougaard family copula

A two dimensional Gumbel-Hougaard copula is defined as

$$C_\theta(u_1, u_2) = \exp(-((-\log u_1)^\theta + (-\log u_2)^\theta)^{1/\theta}), 1 \le \theta \le \infty$$

For $\theta = 1$ the Gumbel-Hougaard copula models independence, for $\theta \to \infty$ it converges to comonotonicity. It is not symmetric and has upper tail dependence.

(1.1c) Applications of copula technique in the present study

With the help of copula technique we can find the joint distribution of random variables following different types of marginal distribution. In the present paper we have applied copula to find the joint probability distribution of repair probabilities when the subsystems A and B are under repair where the repair rates for both the subsystems are different.

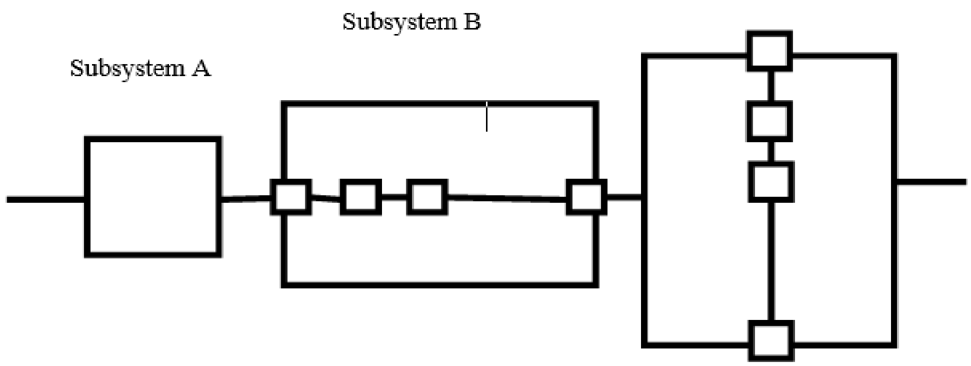

Figure 1: Diagram of investigated system

1.2. Assumptions

(1) Initially the system is in perfectly operating state and subsystems A, B and C are connected in series.

(3) Subsystem A can fail from fully operational state only.

(4) Subsystem B has *m* identical units arranged in series.

(5) In Subsystem C there are two types of units: (i) ordinary and (ii) key, arranged alternatively in the system.

(6) The units having weight 1 are ordinary units while those with weight 2 are called key units.

(7) The subsystem C and hence the whole system will be in degraded and failed states if in C the failed units have weight 3 and 4 respectively.

(8) If more than two key units are in failed state then two different repair facilities will repair the system simultaneously with different rates.

(9) System will be repaired only when three successive failures have occurred in the system or the whole system is in failed state.

(10) After repair the system becomes as good as new.

(11) The joint probability distribution of repairs when two repair facilities are repairing simultaneously the failed units with different rates is given by Gumbel-Hougaard family of copula.

1.3. Acronym

MSS: Multi State System

M.T.T.F.: Mean time to failure

1.4. Nomenclature

λ_A : Failure rate of subsystem A.

λ_B : Failure rate of a single unit of subsystem B.

λ_1 : Failure rate of a single ordinary unit.

λ_2 : Failure rate of a single key unit.

$\phi(x)$: Repair rate of repair facility one.

$\psi(x)$: Repair rate of repair facility two.

x: Elapsed repair time.

$P_i\,(t)$: Probability that the system is in S_i state at instant t for $i = 1$ to $i = 17$.

$\overline{P}_i\,(s)$: Laplace transform of P_i (t).

$P_4(x, t)$: Probability density function that at time t the system is in failed state S_4 and the system is under repair, elapsed repair time is x.

$E_p\,(t)$: Expected profit during the interval (0, t].

K_1, K_2: Revenue per unit time and service cost per unit time respectively.

$S_\eta(x)$: $\eta(x)\exp(-\int_0^x \eta(x)dx)$

$\overline{S}_\eta(x)$: Laplace transform of $S_\eta(x) = \int_0^\infty \eta(x)\exp(-sx - \int_0^x \eta(x)dx)$

If $u_1 = \phi(x)$, $u_2 = \psi(x)$ then the expression for the joint probability according to Gumbel-Hougaard family of copula is given as

$$C_\theta(u_1, u_2) = \exp[-\{(-\log\phi(x))^\theta + (-\log\psi(x))^\theta\}^{1/\theta}]$$

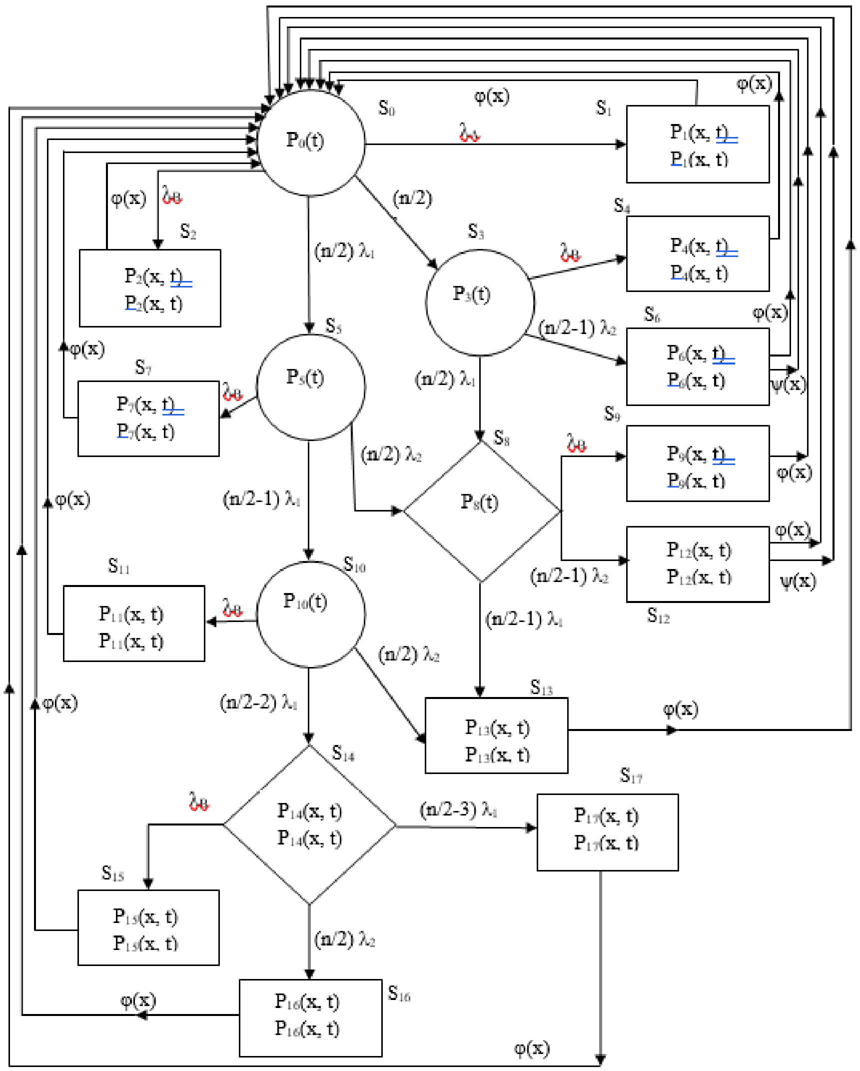

Figure 2: Transition Diagram of proposed system

States	State of subsystem A	Subsystem B: Number of Good units	Subsystem C: Number of good units		Weight of failed units of C	System state
			Type 1	Type 2		
S_0	O	m	$n/2$	$n/2$	0	O
S_1	F	m	$n/2$	$n/2$	0	F
S_2	O	m-1	$n/2$	$n/2$	0	F
S_3	O	m	$n/2$	$n/2$-1	2	O
S_4	O	m-1	$n/2$	$n/2$-1	2	F
S_5	O	m	$n/2$-1	$n/2$	1	O
S_6	O	m	$n/2$	$n/2$-2	4	F
S_7	O	m-1	$n/2$-1	$n/2$	1	F

S_8	O	*m*	*n*/2-1	*n*/2-1	3	D
S_9	O	*m*-1	*n*/2-1	*n*/2-1	3	F
S_{10}	O	*m*	*n*/2-2	*n*/2	2	O
S_{11}	O	*m*-1	*n*/2-1	*n*/2	3	F
S_{12}	O	*m*	*n*/2-1	*n*/2-2	5	F
S_{13}	O	*m*	*n*/2-2	*n*/2-1	4	F
S_{14}	O	*m*	*n*/2-3	*n*/2	3	D
S_{15}	O	*m*-1	*n*/2-3	*n*/2	3	F
S_{16}	O	*m*	*n*/2-3	*n*/2-1	5	F
S_{17}	O	*m*	*n*/2-4	*n*/2	4	F

O: Operational state, F: Failed state, D: Degraded state

Table 1: State Specification Chart

We have formulated our model and then solve it using Laplace transform which gives the up and down state probabilities corresponding to considered system are given by

$$\overline{P}_{up}(s)=\overline{P}_0(s)+\overline{P}_3(s)+\overline{P}_5(s)+\overline{P}_8(s)+\overline{P}_{10}(s)+\overline{P}_{14}(s)$$

$$\overline{P}_{up}(s)=\Bigg[1+\frac{n}{2}\frac{\lambda_2}{\left(s+\lambda_B+\left(\frac{n}{2}-1\right)\lambda_2+\frac{n}{2}\lambda_1\right)}+\frac{n}{2}\frac{\lambda_1}{\left(s+\lambda_B+\left(\frac{n}{2}-1\right)\lambda_1+\frac{n}{2}\lambda_2\right)}\Bigg\{1+\frac{n}{2}\times$$

$$\times\frac{\lambda_2}{\left(s+\lambda_B+\left(\frac{n}{2}-1\right)\lambda_2+\left(\frac{n}{2}-1\right)\lambda_1\right)}+\frac{\left(\frac{n}{2}-1\right)\lambda_1}{s+\lambda_B+\frac{n}{2}\lambda_2+\left(\frac{n}{2}-2\right)\lambda_1}$$

$$+\frac{\left(\frac{n}{2}-2\right)\lambda_1\left(\frac{n}{2}-1\right)\lambda_1}{s+\lambda_B+\frac{n}{2}\lambda_2+\left(\frac{n}{2}-2\right)\lambda_1}\frac{1-\overline{S}_\phi\left\{s+\left(\frac{n}{2}-3\right)\lambda_1+\lambda_B+\frac{n}{2}\lambda_2\right\}}{s+\left(\frac{n}{2}-3\right)\lambda_1+\lambda_B+\frac{n}{2}\lambda_2}\Bigg\}\Bigg]\frac{1}{D(s)} \qquad \ldots(1)$$

$$\overline{P}_{down}(s)=\overline{P}_1(s)+\overline{P}_2(s)+\overline{P}_4(s)+\overline{P}_6(s)+\overline{P}_7(s)+\overline{P}_9(s)+\overline{P}_{11}(s)+\overline{P}_{12}(s)+\overline{P}_{13}(s)$$
$$+\overline{P}_{15}(s)+\overline{P}_{16}(s)+\overline{P}_{17}(s)$$

$$\overline{P}_{down}(s)=\Bigg[(\lambda_A+\lambda_B)\frac{1-\overline{S}_\phi(s)}{s}+\frac{\frac{n}{2}\lambda_2}{\left(s+\lambda_B+\left(\frac{n}{2}-1\right)\lambda_1+\frac{n}{2}\lambda_2\right)}\Bigg\{\lambda_B\frac{1-\overline{S}_\phi(s)}{s}+\left(\frac{n}{2}-1\right)\times$$

$$\times\lambda_2\frac{1-\overline{S}_\xi(s)}{s}\Bigg\}+\Bigg\{\frac{n}{2}\frac{\lambda_2}{\left(s+\lambda_B+\left(\frac{n}{2}-1\right)\lambda_2+\left(\frac{n}{2}-1\right)\lambda_1\right)}\Bigg\{\lambda_B\frac{1-\overline{S}_\phi(s)}{s}+\left(\frac{n}{2}-1\right)\times$$

$$\times\lambda_2\frac{1-\overline{S}_\xi(s)}{s}\Bigg\}+\frac{\left(\frac{n}{2}-1\right)\lambda_1}{\left(s+\lambda_B+\frac{n}{2}\lambda_2+\left(\frac{n}{2}-2\right)\lambda_1\right)}\Bigg\{\lambda_B\frac{1-\overline{S}_\phi(s)}{s}+\frac{n}{2}\lambda_2\frac{1-\overline{S}_\phi(s)}{s}$$

$$+\left(\frac{n}{2}-2\right)\lambda_1\frac{1-\overline{S}_\phi\left\{s+\left(\frac{n}{2}-3\right)\lambda_1+\lambda_B+\frac{n}{2}\lambda_2\right\}}{s+\left(\frac{n}{2}-3\right)\lambda_1+\lambda_B+\frac{n}{2}\lambda_2}\Bigg\}+\left(\frac{n}{2}-2\right)\lambda_1\Bigg\{\Bigg\{\frac{1-\overline{S}_\phi(s)}{s}\Bigg\}$$

$$-\frac{1-\overline{S}_\phi\left\{s+\left(\frac{n}{2}-3\right)\lambda_1+\lambda_B+\frac{n}{2}\lambda_2\right\}}{s+\left(\frac{n}{2}-3\right)\lambda_1+\lambda_B+\frac{n}{2}\lambda_2}\Bigg\}\frac{\frac{n}{2}\lambda_1}{s+\frac{n}{2}\lambda_2+\lambda_B+\left(\frac{n}{2}-1\right)\lambda_1}\Bigg]\overline{P}_0(s) \qquad \ldots(2)$$

From (1) and (2), we have

$$\overline{P}_{\text{up}}(s)+\overline{P}_{\text{down}}(s)=1/s \qquad \ldots(3)$$

1.5. Numerical computation

To numerically solve the model, failure rates and some other parameters are fixed as

$$\lambda_A=0.10, \lambda_B=0.20, \lambda_1=0.10, \lambda_2=0.10, n=6, \theta=1, x=1 \qquad \ldots(4)$$

to obtain the reliability, availability, M. T. T. F. and cost analysis of the system. Also let the repairs are following exponential distribution.

(1.5a) Availability analysis

To determine availability, in addition to all values given in equation (4), we put $\Phi = \psi = 1$ in equation (1) and taking inverse Laplace transform, one can obtain

$$P_{up}(t)= -0.02340590215\, e^{(-1.968057682\, t)} + 0.2273962706\, e^{(-1.529358993\, t)} \cos(0.06335397435\, t) +0.4478794215\, e^{(-1.529358993\, t)} \sin(0.06335397435\, t)+.1021738894\, e^{(-0.8422478639\, t)} \cos(0.4610873718\, t)+.1296843609\, e^{(-0.8422478639\, t)} \sin(0.4610873718\, t)- 0.007734987489\, e^{(-0.6806418758\, t)} +0.7015707297 e^{(-0.008086729754\, t)} \qquad \ldots(5)$$

Varying *t* from 0 to 10 in equation (5), one can obtain the variation of availability with respect to time. Table 2 and Fig. 3 are corresponding to availability analysis of the system.

(1.5.b) Reliability Analysis

For this we take repairs $\Phi = \psi = 0$ and the values mentioned in equation (4). Now by setting $t = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, one can obtain Table 3 and correspondingly Fig. 4 which represents the behaviour of reliability as time increases.

(1.5.c) M.T.T.F. Analysis

The M.T.T.F. of the system can be calculated with the help of the formula given below:

$$\text{M.T.T.F.} = \lim_{s\to 0} \overline{P}_{\text{up}}(s)$$

With the condition $\Phi = \psi = 0$. Variation of M.T.T.F. of the system with respect to each failure rate gives following four cases.

(a) Varying λ_A from 0.10 to 0.90 and assuming all the other values as given in equation (4), one can obtain Table 4 which demonstrates variation of M.T.T.F. with respect to λ_A.

(b) Increasing the value of λ_B from 0.10 to 0.90 and taking the values of other parameters as given in equation (4), we get Table 5 which shows how M.T.T.F. changes with respect to λ_B.

(c) When $\lambda_1 = 0.10, 0.20, 0.30, 0.40, 0.50, 0.60, 0.70, 0.80, 0.90$ and all the other parameters have the values as in equation (4), we compute Table 6 which represents this variation of M.T.T.F. with the increasing value of λ_1.

(d) Increase the value of λ_2 from 0.10 to 0.90 and assume the other parameters have the values as given in equation (4); we obtain Table 7 which represents the manner in which M.T.T.F. varies with respect to λ_2.

Variation of M.T.T.F with respect to λ_A, λ_B, λ_1 and λ_2 in the above four cases (a), (b), (c) and (d) have been given in the Figs. 5, 6, 7 and 8 respectively.

(1.5.d) Cost Analysis

If the service facility is always available, then expected profit during the interval $(0, t]$ is given by $E_P(t) = K_1 \int_0^t P_{up}(t)dt - K_2 t$ where K_1 and K_2 are the revenue per unit time and service cost per unit time respectively, then by using the expression of $P_{up}(t)$ given in equation (1), one can obtain

$$\begin{aligned} E_P(t) = K_1[&0.01189289438\, e^{(-1.968057682\, t)} - 0.1605433520\, e^{(-1.529358993\, t)} \cos(0.06335397435\, t) \\ &-0.2862038044\, e^{(-1.529358993\, t)} \sin(0.06335397435\, t) - 0.1581933254 e^{(-0.8422478639\, t)} \\ &\cos(0.4610873718\, t) - 0.06737139827\, e^{(-0.8422478639\, t)} \sin(0.4610873718\, t) \\ &+0.01136425448\, e^{(-0.6806418758\, t)} - 86.75580254\, e^{(-0.008086729754\, t)} + 87.05128207] - K_2 t \end{aligned} \quad ...(6)$$

Keeping $K_1 = 1$ and varying K_2 as 0.1, 0.2, 0.3, 0.4, 0.5 in equation (6), one can obtain Table 8 and correspondingly Fig. 9.

Time	P_{up}
0	1.0000000000
1	0.8083176983
2	0.7317495264
3	0.6985347132
4	0.6828038217
5	0.6740921819
6	0.6679476953
7	0.6625921939
8	0.6574056828
9	0.6522235942
10	0.6470308620

Table 2: Time vs. Availability

Time	P_{up}
0	1.0000000000
1	0.7127091428
2	0.4789341661
3	0.3090432700
4	0.1936063458
5	0.1186016147
6	0.07139680698
7	0.04238641267
8	0.02488164142
9	0.01447130279
10	0.00835199693

Table 3: Time vs. Reliability

λ_A	MTTF
.10	2.460317460
.20	2.214285713
.30	2.012987013
.40	1.845238095
.50	1.703296703
.60	1.581632653
.70	1.476190476
.80	1.383928571
.90	1.302521008

Table 4: λ_A vs. M.T.T.F.

λ_B	MTTF
.10	3.125000000
.20	2.460317460
.30	2.017857143
.40	1.704545455
.50	1.472222222
.60	1.293706294
.70	1.152597403
.80	1.038461538
.90	0.944368132

Table 5: λ_B vs. M.T.T.F

λ_1	MTTF
.10	2.460317460
.20	2.164682540
.30	1.938694639
.40	1.756588320
.50	1.605800215
.60	1.478665330
.70	1.369980507
.80	1.276007326
.90	1.193964563

Table 6: $\lambda_1$vs. M.T.T.F.

λ_1	MTTF
.10	2.460317460
.20	1.881944444
.30	1.530069930
.40	1.290598291
.50	1.116481681
.60	0.983993029
.70	0.879727095
.80	0.795502645
.90	0.726032478

Table 7: λ_2 vs. M.T.T.F.

Time	$E_P(t)$				
	K_2=0.1	K_2=0.2	K_2=0.3	K_2=0.4	K_2=0.5
0	0	0	0	0	0
1	0.78905866	0.68905866	0.58905866	0.48905866	0.38905866
2	1.45361363	1.25361363	1.05361363	0.85361363	0.65361363
3	2.06657323	1.76657323	1.46657323	1.16657323	0.86657323
4	2.65634931	2.25634931	1.85634931	1.45634931	1.05634931
5	3.23445604	2.73445604	2.23445604	1.73445604	1.23445604
6	3.80536187	3.20536187	2.60536187	2.00536187	1.40536187
7	4.37060253	3.67060253	2.97060253	2.27060253	1.57060253
8	4.93059803	4.13059803	3.33059803	2.53059803	1.73059803
9	5.48541394	4.58541394	3.68541394	2.78541394	1.88541394
10	6.03504139	5.03504139	4.03504139	3.03504139	2.03504139

Table 8: Time vs. expected profit

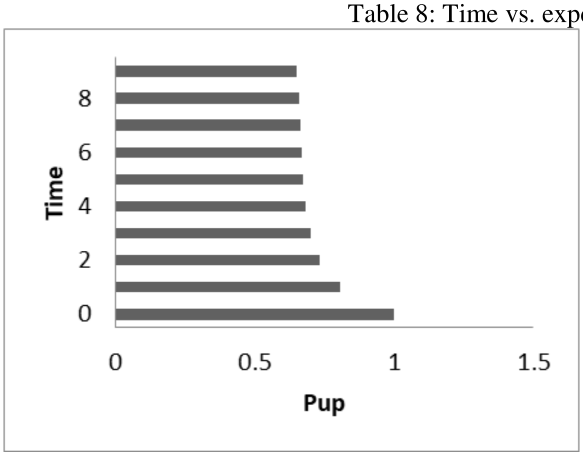

Figure 3: Time vs. Availability

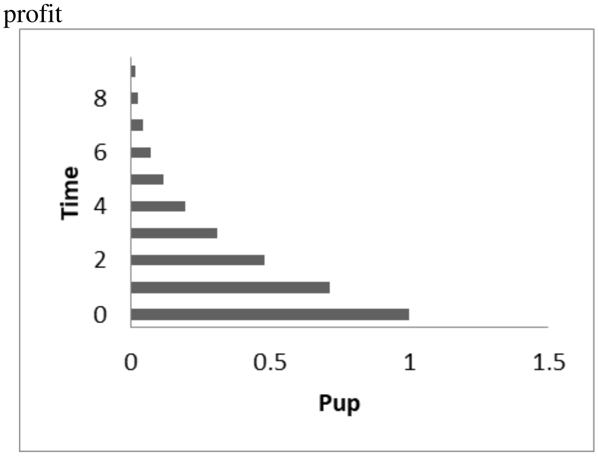

Figure 4: Time vs. Reliability

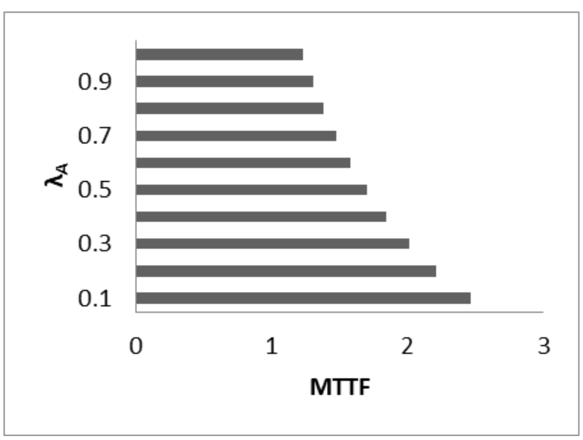

Figure 5: λ_A vs. M.T.T.F.

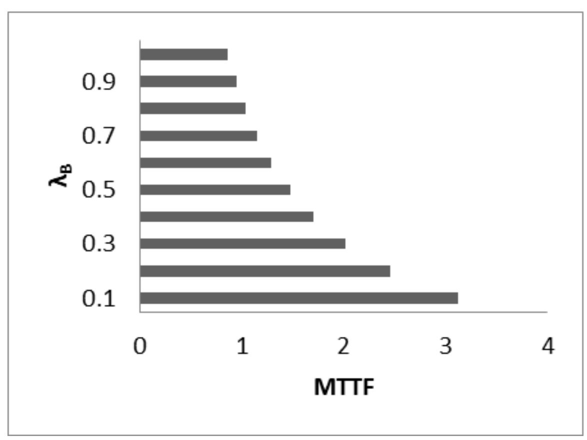

Figure 6: λ_B vs. M.T.T.F.

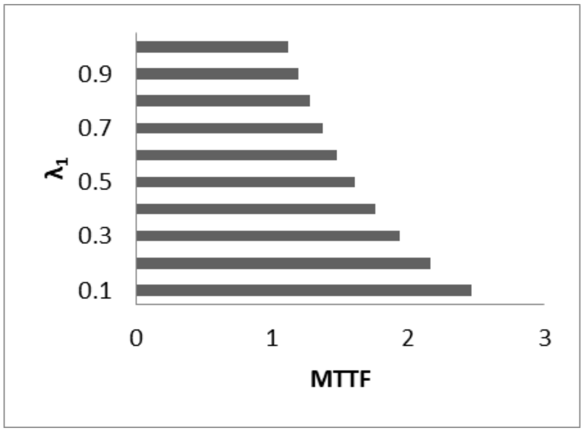

Figure 7: λ_1 vs. M.T.T.F.

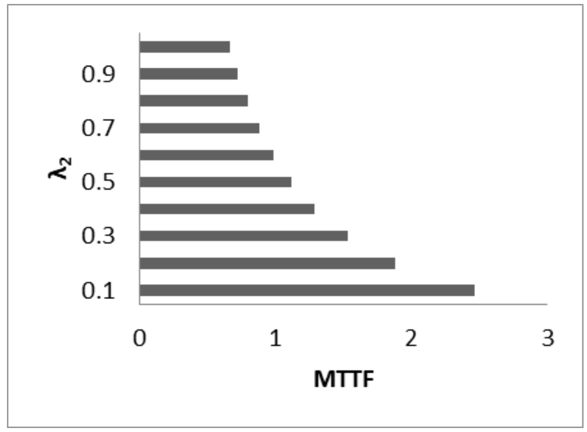

Figure 8: λ_2 vs. M.T.T.F.

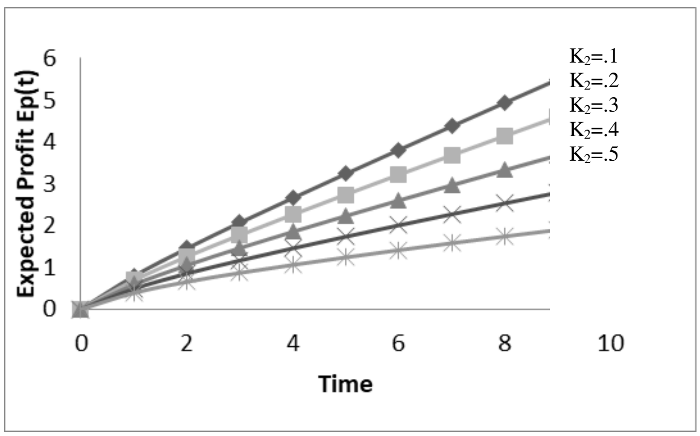

Figure 9: Time vs. expected profit

1.6. Conclusions

Here in the present study different characteristics such as transition state probabilities, availability, reliability, M.T.T.F., cost analysis of a system having three subsystems have been analysed. For numerical computation we have fixed $n = 6$, $\lambda_A = 0.10$, $\lambda_B = 0.20$, $\lambda_1 = 0.10$ and $\lambda_2 = 0.10$. Fig. 3 shows that availability of the system decreases with the increment in the time and later on it stabilizes at a value 0.6. Again the reliability of the system decreases as time increases as shown in Fig. 4. Initially at time $t = 0$ both reliability and availability of the system has value 1.

As far as the M.T.T.F. of the system is concerned, we can observe from Figs. 5, 6, 7 and 8 which are corresponding to variation in M.T.T.F. with respect to λ_A, λ_B, λ_1 and λ_2 respectively that in each case M.T.T.F. of the system decreases with the increment in failure rate. M.T.T.F. varies from 2.46-1.30, from 3.12-0.94, from 2.46-1.19 and from 2.46-0.79 in the cases of λ_A, λ_B, λ_1 and λ_2 respectively.

For the cost analysis of the system we keep revenue cost per unit time at 1 and vary service cost from 0.1 to 0.5. From Fig. 9 one can conclude that increasing service cost results decrement in the expected profit of the system. The highest and lowest values of expected profit are 6.03 and 0.38 respectively for the considered values.

1.7. References

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