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Numerical Simulation of Fifth order Linear and Nonlinear BVPs via Haar wavelet method

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Abstract

In this research article, the numerical solution of fifth order boundary value problems (BVPs) is presented. For this purpose, application of the Haar wavelet method has been discussed. In this method, space derivative terms are approximated using truncated Haar wavelet series. The present method is tested on fifth order linear and nonlinear BVPs. The accuracy of the proposed technique is demonstrated through two numerical examples, taken from existing literature. The present approximate results are shown through tables and figures. The numerical solutions demonstrate the high accuracy of the present method.

Keywords and Phrases. Boundary Value Problem, Haar Wavelet Method, Linear and Nonlinear Differential Equation, Convergence Analysis.

1. INTRODUCTION

BVPs (BVPs) of higher order play a significant role in various fields of science and engineering, particularly when modeling complex physical phenomena. Among these, fifth-order BVPs arise in a range of applications, including fluid mechanics, elasticity theory, and the analysis of nonlinear dynamical systems. These equations also arise in the mathematical modelling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences [1-4]. Due to the complexity and the nature of the underlying differential equations, obtaining accurate and reliable solutions for these problems poses a significant challenge.

Over the past few decades, numerous analytical and numerical techniques have been developed to address fifth-order BVPs. Classical methods such as the finite difference method (FDM), finite element method (FEM), and variational techniques have been widely used to find approximate solutions [5-6]. However, these traditional approaches may suffer from computational inefficiency or lack of accuracy when dealing with highly nonlinear or complex boundary conditions. Recent advancements in numerical techniques have introduced more robust methods to solve fifth-order BVPs. B-spline method [1], and variational iteration methods [6] have gained prominence due to their ability to handle complex boundary conditions and nonlinearity effectively.

Over the past two decades, wavelets have gained significant attention in the field of numerical estimation and have found numerous applications in estimation theory [7, 8]. The literature reflects a wealth of research utilizing wavelets for solving ordinary differential equations (ODEs), integral equations (IEs), PDEs, numerical integration (NI), and fractional partial differential equations (FPDEs) [9-12]. Among the various types of wavelets, the Haar Wavelet (HW) has received particular interest due to its simplicity and effectiveness in solving integral and differential equations [13-15]. Effective compression and analysis are made possible by the Haar wavelet approach, which is useful in everyday applications like image processing, biomedical signal processing, face recognition, and fingerprint compression. It excels in solving intricate mathematical problems in numerical analysis, especially in everyday life, its numerous uses highlight how important it is to the advancement of technology and computing efficiency [16]. Because they make multiresolution analysis and sparse signal representation possible, Haar wavelets are essential to numerical solutions. They are useful tools for solving differential equations, filtering noise, and handling a variety of numerical problems because of their localization qualities in time and frequency, flexibility in handling irregular grids, and compression efficiency. In summary, Haar wavelets help numerical solutions be more accurate and computationally efficient. The simplest wavelet family with a compact support is the Haar wavelet family, which consists of orthonormal wavelets distinguished by their piecewise constant functions. This simplicity has been useful in solving higher-order differential and integral equations and allows for effective numerical approximations [17, 18]. However, using Haar wavelets directly to differential equations presents difficulties due to their piecewise constant character. Chen and Hsiao [19, 20] presented an integrated strategy to address issue, and Lepik [21, 22] expanded on it.

The fundamental idea behind this method is to convert a differential equation into an algebraic one. The collocation points are used to the resulting algebraic equations in order to estimate the solution of the differential equation. One of the main benefits of the Haar wavelet transform is that, in comparison to other current techniques, it produces a sparse matrix representation, which increases efficiency.

This paper aims to explore the application of wavelet-based methods for solving fifth-order BVPs. The proposed approach not only improves computational efficiency but also maintains a high degree of accuracy, as demonstrated through several numerical examples drawn from existing literature. The general fifth order boundary value problem (BVP) under consideration are as follows:

$$u^{(v)}(x) = f\left(x, u(x), u'(x), u''(x), u'''(x), u^{(iv)}(x)\right), \quad x \in [a, b]. \tag{1}$$

with boundary conditions

$$u(a) = A_1, \quad u'(a) = A_2, \quad u''(a) = A_3, \quad u(b) = B_1, \quad , \quad u'(b) = B_2, \tag{2}$$

where, $u(x)$ and $f(x)$ are continuous functions defined in the interval $x \in [a, b]$, and A_1, A_2, A_3, B_1 and B_2 are finite real constants. Generally, this type of fifth order BVPs arises in the mathematical modeling of viscoelastic flows and other branches of mathematical, physical and engineering sciences.

The rest of the paper is organized as follows: In section 2, basic preliminaries of Haar wavelets are given. In section 3, general formulation of the present numerical method is given. Section 4 presents convergence analysis of Haar wavelet. Next, two numerical problems for comparison with current methods are presented just before the last section. Lastly, a conclusion is provided in the final section.

2. HAAR WAVELETS

In the area of approximation theory, Haar wavelets has several uses. Because of their piecewise constant character, these wavelets are very good at capturing abrupt discontinuities or transitions in functions. Due to this feature, they are helpful in situations where there are discontinuous solutions, like in signal processing or specific differential equations. The sparse representation of these wavelets allows them to express functions with fewer coefficients more effectively than other bases, such as polynomial or Fourier bases. This can result in methods that are more computationally efficient, particularly when handling high-dimensional problems or big datasets.

Additionally, because of piecewise constant nature of Haar wavelets, Haar wavelet-based method has strong numerical stability, is simple to implement, and provides a practical method of handling boundary conditions. It is important to remember that, despite these benefits, efficacy of Haar wavelet method varies depending on the specific characteristics of the problem. The particulars of the problem, the computational resources at hand, and the trade-offs between accuracy, efficiency, and implementation complexity are frequently taken into consideration while selecting a solution.

The family of Haar wavelets over [0,1] is given by

$$h_i(\varphi) = \begin{cases} 1 & \varphi \in [\varphi_1, \varphi_2), \\ -1 & \varphi \in [\varphi_2, \varphi_3), \\ 0 & elsewhere \end{cases} \qquad i = 2,3, \ldots \tag{3}$$

where,

$\varphi_1 = \frac{k}{m}$, $\varphi_2 = \frac{k+0.5}{m}$, $\varphi_3 = \frac{k+1}{m}$.

Here, integer $m = 2^p, p = 0,1, \ldots, J$, where J is the maximum resolution level and translation parameter $k = 0,1, \ldots, m-1$. The relationship between i, m and k is given by $i = m + k + 1$ such that $2 \le i \le 2M$, where $M = 2^J$.

The Haar wavelet scaling functions can be described as

$$h_1(\varphi) = \begin{cases} 1 & \varphi \in [0,1) \\ 0 & elsewhere. \end{cases} \tag{4}$$

An infinite sum of Haar wavelets can be used to express any sparsely integrable function in the interval (0,1) as follows:

$$f(\varphi) = \sum_{i=1}^{\infty} a_i h_i(\varphi).$$

If $f(\varphi)$ is piecewise constant or roughly piecewise constant across each subinterval, the aforementioned series ends at finite term given as follows:

$$f(\varphi) = \sum_{i=1}^{M} a_i h_i(\varphi).$$

The successive integrals of Haar wavelet functions are

$$p_{i,1}(\varphi) = \int_0^x h_i(\varphi) d\varphi,$$

and

$$\left.\begin{aligned} p_{i,n+1}(\varphi) &= \int_0^x p_{i,n}(\varphi)d\varphi, \\ C_{i,n} &= \int_0^1 p_{i,n}(\varphi)d\varphi \end{aligned}\right\}, n = 1,2,3,\ldots \quad (8)$$

Using expressions (7) and (8), n^{th} integral of Haar wavelet functions is given by

$$p_{1,n}(\varphi) = \frac{\varphi^n}{n!}, n = 1,2,3,.. \quad (9)$$

and

$$p_{i,n}(\varphi) = \begin{cases} \frac{(x-\varphi_1)^n}{n!}, & x \in [\varphi_1, \varphi_2) \\ \frac{(x-\varphi_1)^n}{n!} - 2\frac{(x-\varphi_2)^n}{n!}, & x \in [\varphi_2, \varphi_3) \\ \frac{(x-\varphi_1)^n}{n!} - 2\frac{(x-\varphi_2)^n}{n!} + \frac{(x-\varphi_3)^n}{n!}, & x \in [\varphi_3, 1) \\ 0, & elsewhere. \end{cases} \quad i = 2,3,\ldots\ldots; n = 1,2,3,\ldots \quad (10)$$

3. HAAR WAVELET METHOD FOR SOLVING FIFTH ORDER DIFFERENTIAL EQUATIONS

Here, the present method has been developed which is based on Haar wavelet method. The space derivatives in equation (1) are discretized using Haar wavelet series which is presented in the following manner:

$$u^{(v)}(x) = \sum_{i=1}^{2M} a_i h_i(x). \quad (11)$$

Taking integration and using the boundary conditions with a=0, b=1, the space derivatives $u^{(v)}(x), u^{(iv)}(x), u'''(x), u''(x), u'(x)$ and $u(x)$ are obtained which are given as:

$$u^{(iv)}(x) = u^{(iv)}(0) + \sum_{i=1}^{2M} a_i p_{i,1}(x),$$

$$u'''(x) = u'''(0) + xu^{(iv)}(0) + \sum_{i=1}^{2M} a_i p_{i,2}(x),$$

$$u''(x) = u''(0) + xu'''(0) + \frac{x^2}{2}u^{(iv)}(0) + \sum_{i=1}^{2M} a_i p_{i,3}(x), \quad (14)$$

$$u'(x) = u'(0) + xu''(0) + \frac{x^2}{2}u'''(0) + \frac{x^3}{6}u^{(iv)}(0) + \sum_{i=1}^{2M} a_i p_{i,4}(x), \quad (15)$$

$$u(x) = u(0) + xu'(0) + \frac{x^2}{2}u''(0) + \frac{x^3}{6}u'''(0) + \frac{x^4}{24}u^{(iv)}(0) + \sum_{i=1}^{2M} a_i p_{i,5}(x), \qquad (16)$$

where, $p_{i,1}$, $p_{i,2}$, $p_{i,3}$, $p_{i,4}$ and $p_{i,5}$ are obtained using equations (9) and (10) respectively.

The value of unknown terms $u'''(0)$ and $u^{(iv)}(0)$ are calculated by integrating equations (14) and (15) from 0 and 1 and is given by

$$u'''(0) = 24\big(u(1) - u(0)\big) - 18u'(0) - 6u'(1) - 6u''(0) - 24\sum_{i=1}^{2M} a_i p_{i,5}(1) + 6\sum_{i=1}^{2M} a_i p_{i,4}(1),$$

$$u^{(iv)}(0) = 72\big(-u(1) + u(0)\big) + 48u'(0) + 24u'(1) + 12u''(0) - 24\sum_{i=1}^{2M} a_i p_{i,4}(1) + 72\sum_{i=1}^{2M} a_i p_{i,5}(1). \qquad (18)$$

These values are used to create a system of equations, the solution of which provides the Haar coefficients, by substituting them in the expressions (12), (13), (14), (15), and (16). The degree of resolution of Haar wavelets is inversely related to the error bound, as demonstrated by Babolian and Shahsavaran [17]. When M is increased, this guarantees that the Haar wavelet approximation will converge.

4. CONVERGENCE THEOREM

Theorem 1 Assume that $g(x) = u^n(x) \in L^2(R)$ is a continuous function on [0, 1] with a bounded first derivative:

$$\forall\, x \in [0,1], \qquad \exists\, \sigma: |g'(x)| \le \sigma, \quad \sigma \ge 2.$$

Then, the Haar wavelet will be convergent, i.e.,$|E_m|$ vanishes as J goes to infinity, according to the method suggested in [23]. This convergence is of order two:

$$\|E_m\|_2 = O\left[\left(\frac{1}{2^{J+1}}\right)^2\right], \qquad (17)$$

where,

$$E_m = g(x) - g_M(x), \quad g_M(x) = \sum_{i=1}^{2M} a_i h_i(x).$$

Proof For proof, see [23].

5. NUMERICAL EXAMPLES

To evaluate the effectiveness of the method, both linear and nonlinear fifth order BVPs are considered. The accuracy of the method has been assessed by evaluating the absolute error, maximum absolute errors. These errors will be computed using a varying number of collocation points. Here, MATLAB R2015a software has been used for numerical computation.

The absolute error, maximum absolute error and L_2error norm will be denoted by

Absolute error $\|e_J\| = |u(x_i) - u_J(x_i)|$,

$L_\infty = \max_i |u(x_i) - u_J(x_i)|$,

Example 5.1: Consider the linear boundary value problem

$$u^v(x) - u(x) = -15e^x - 10xe^x$$
$$= 0, \quad 0 < x < 1.$$

with boundary conditions

$u(0) = 0,\ u'(0) = 1,\ u''(0) = 0,\ u(1) = 1, u'(1) = -e.$

The exact solution for the above example is $x(1-x)e^x$.

The proposed method is applied to this example, and the corresponding numerical results are displayed in both tabular and graphical formats. To evaluate the accuracy, absolute and maximum absolute errors are computed for various numbers of collocation points. For the given test problem, the maximum absolute error has been effectively minimized to the order of 10^{-8}, which is generally acceptable for practical applications. It is worth noting that even greater accuracy can be attained by increasing the number of collocation points; however, this improvement comes with a trade-off in terms of higher computational cost and time. A key advantage of the Haar wavelet method is its ability to yield increasingly accurate solutions as the number of collocation points rises.

Table 1

Comparison of absolute error in results obtained by present method and other method at *J=4* for Example 1

x	Exact Solution	Variational Iteration Method [6]	Sixth degree B-Spline Method [1]	Haar Wavelet Method
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.0995	0.188E-4	-8.0E-3	1.58E-08
0.2	0.1954	1.077E-4	-1.2E-3	1.02E-07
0.2	0.2835	2.477E-4	-5.0E-3	2.70E-07

0.4	0.3580	3.729E-4	3.0E-3	4.82E-07
0.5	0.4122	4.202E-4	8.0E-3	6.68E-07
0.6	0.4373	3.643E-4	6.0E-3	7.56E-07
0.7	0.4229	2.364E-4	-0.000	6.91E-07
0.8	0.3561	1.158E-4	9.0E-3	4.70E-07
0.9	0.2214	0.876E-4	-9.0E-3	1.71E-07
1.0	0.0000	0.0000	0.0000	0.0000

Table 2

Maximum absolute error for Example 1 at different number of collocation points

J	$2M$	L_∞
1	4	4.6102E-05
2	8	1.1612E-05
3	16	3.0085E-06
4	32	7.5712E-07
5	64	1.8937E-07
6	128	4.7370E-08
7	256	1.1844E-08

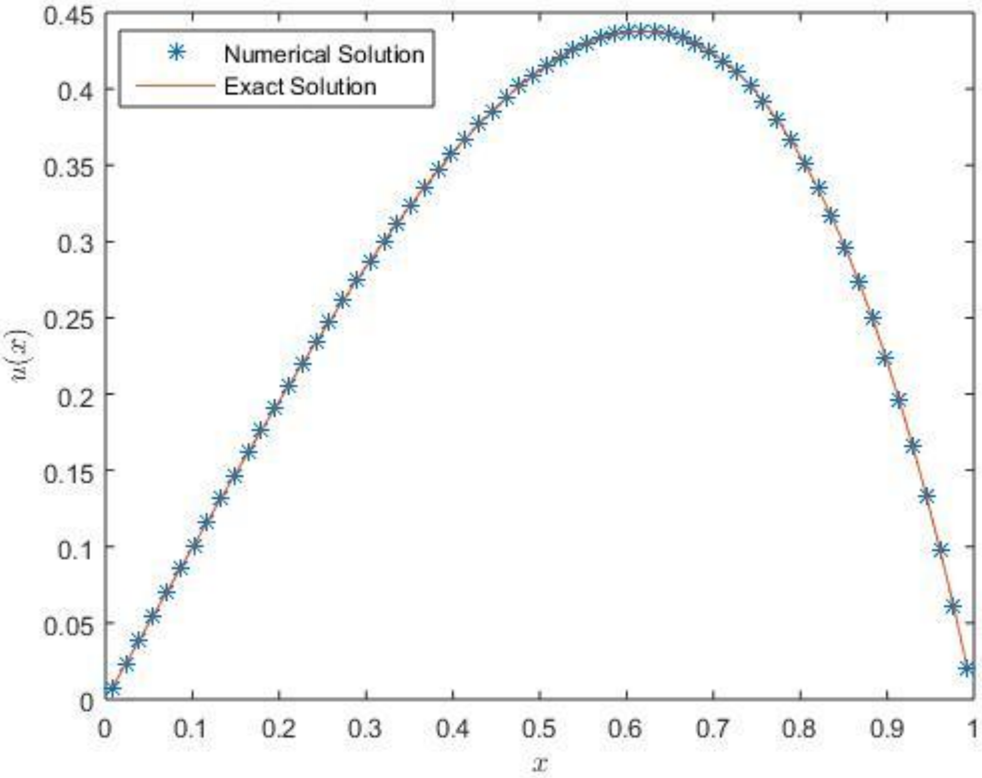

Figure 1. Comparison of the exact solutions and numerical solutions of Example 1

Example 5.2: Consider the no-linear boundary value problem

$$u^{v}(x) = e^{-x}u^{2}(x), \quad 0 < x < 1.$$

with boundary conditions

$u(0) = 1, u'(0) = 1, u''(0) = 1, u(1) = e, u'(1) = e.$

The exact solution for the above example is e^{x}.

In this case study, the proposed method has been applied to solve the given problem. To address the nonlinear nature of the equation, the quasilinearization technique is adopted to convert it into a linear form. The accuracy of the obtained solution is examined by calculating the absolute error and the maximum absolute error for different numbers of collocation points. The computed results are illustrated through tables and graphs. From the table, it is evident that the maximum absolute error decreases as the number of collocation points increases, demonstrating that higher numbers of collocation points contribute to greater accuracy. Additionally, Figure 2 visually compares the exact and numerical solutions for various values of x.

Table 3

Comparison of absolute error in results obtained by present method and other method at *J=4* for Example 2

x	Exact Solution	Variational Iteration Method [6]	Sixth degree B-Spline Method [1]	Haar Wavelet Method
0.0	1.0000	0.0000	0.0000	0.0000
0.1	1.1052	0.0000	-7.0E-4	3.92E-10
0.2	1.2214	1.0E-5	-7.2E-4	2.53E-09
0.2	1.3499	1.0E-5	4.1E-4	6.65E-09
0.4	1.4918	1.0E-4	4.6E-4	1.18E-08
0.5	1.6487	3.2E-4	4.7E-4	1.63E-08
0.6	1.8221	3.6E-4	4.8E-4	1.83E-08
0.7	2.0138	-1.4E-4	3.9E-4	1.66E-08
0.8	2.2255	-3.1E-4	3.1E-4	1.12E-08
0.9	2.4596	-5.8E-4	1.6E-4	4.07E-09
1.0	2.7183	-9.9E-5	0.0000	0.0000

Table 4 Maximum absolute error for Example 2 at different number of collocation points

J	$2M$	L_∞
1	4	1.0564E-06
2	8	2.7781E-07
3	16	7.2690E-08
4	32	1.8326E-08
5	64	4.5864E-09
6	128	1.1475E-09
7	256	2.8692E-10

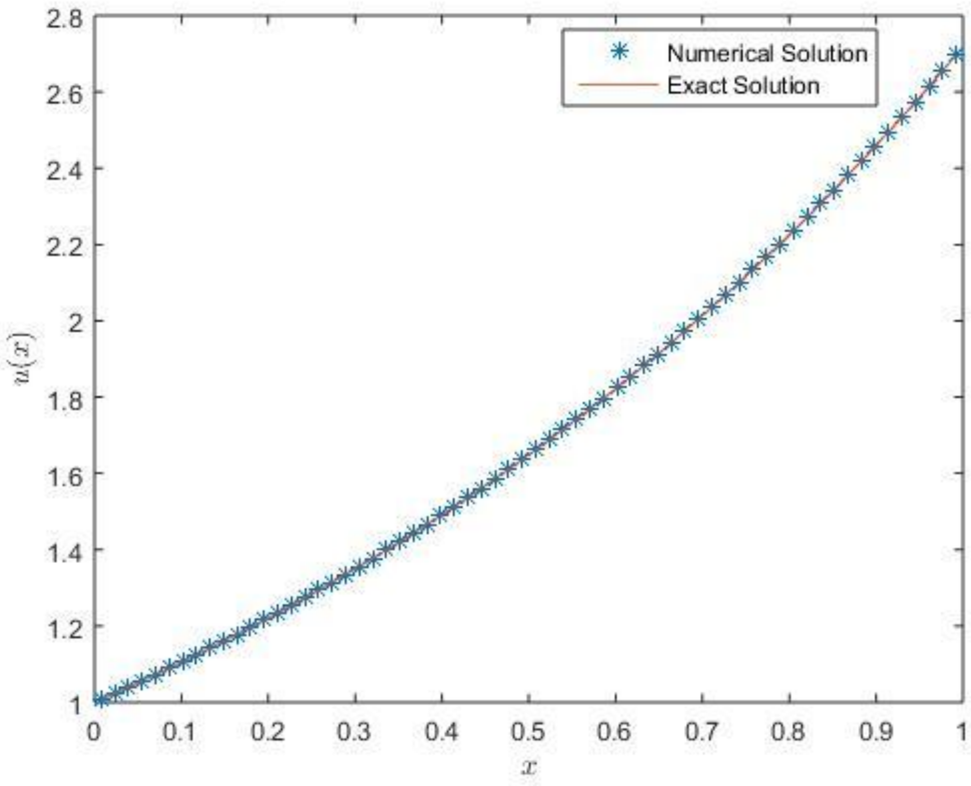

Figure 2. Comparison of the exact solution with the numerical solution of Example 2

6. CONCLUSION

This research paper presents a numerical technique based on the Haar wavelet method for solving both linear and nonlinear fifth-order BVPs. In the case of the linear problem, the Haar wavelet method is directly applied, whereas for the nonlinear problem, the quasilinearization technique is first used to linearize the equation, followed by the application of the Haar wavelet method. The approach is tested on two benchmark problems. Numerical results are compared with exact solutions for varying numbers of collocation points. The method demonstrates high accuracy, achieving a maximum absolute error on the order of 10^{-8} which is considered sufficient for most practical applications. Although increasing the number of collocation points can enhance accuracy further, it also leads to higher computational costs. A notable strength of the proposed method is its ability to reduce maximum absolute error as the number of collocation points increases. Through this analysis, the study establishes that the method is stable, reliable, and effective in capturing sharp variations in the solution.

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